

REMOVABLE SETS FOR HARMONIC
FUNCTIONS
IN BESOV SPACES

By DAVID KIGHURADZE

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DAVID KIGHURADZE

Bachelor of Science
Tbilisi State University
Tbilisi, Georgia
1984

Master of Science
Tbilisi State University
Tbilisi, Georgia
1988

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Graduate College of the
Oklahoma State University
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REMOVABLE SETS FOR HARMONIC FUNCTIONS IN
BESOV SPACES

Thesis Approved:

Dr. David Ullrich

Thesis Advisor

Dr. Dale Alspach

Dr. Alan Noell

Dr. Igor Pritsker

Dr. Kaladi Babu

Dr. Gordon Emslie

Dean of the Graduate College

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1 Introduction

1.1 History of The Problem

Riemann's theorem about removable singularities states that if f is holomorphic in $\Omega \setminus \{a\}$ (Ω is an open set in \mathbb{C}) and f is bounded in $D'(a, r)$ (the punctured disk with radius r and center a) for some $r > 0$, then f has a removable singularity at a . Riemann's theorem raises a question: How can we characterize removable singularities for “good” (analytic, harmonic, etc.) functions?

In 1945 L. Ahlfors [1] introduced the set function γ (“analytic capacity”), and he proved that a compact set K “is removable for bounded analytic functions”, (that is for all open sets $U \supset K$ and all bounded analytic functions f on $U \setminus K$) if and only if $\gamma(K) = 0$.

The definition of γ is purely analytic. So the question arises: What is a geometric characterization of sets with vanishing analytic capacity?

It is not difficult to show that if compact K is thick enough, i.e. if its Hausdorff dimension $\dim(K)$ is strictly more than 1 then $\gamma(K) > 0$. If K is too small, i.e. $\dim(K) < 1$ (then of course $m_1(K) = 0$) or even $\dim(K) = 1$ but $m_1(K) = 0$ then $\gamma(K) = 0$ (m_β denotes the β -dimensional Hausdorff measure of K).

In 1959 A. G. Vitushkin [17] gave an example of a compact set K such that $m_1(K) > 0$ but $\gamma(K) = 0$ (Vitushkin's example was quite complicated but J. Garnett [6] and L. D. Ivanov [8] found a much simpler one).

In 1967 Vitushkin conjectured that $\gamma(K) = 0$ if and only if $m_1(\pi_\theta(K)) = 0$ for almost every direction θ , where π_θ denotes the orthogonal projection from the plane to the line $L_\theta = \{x + iy : x \cos(\theta) + y \sin(\theta) = 0\}$.

Vitushkin's conjecture was confirmed [3] in case when $0 < m_1(K) < \infty$.

In case when $m_1(K) = \infty$, P. Mattila [10], P. Jones and T. Murai [9] showed that Vitushkin's conjecture is not right.

M. Melnikov conjectured in 1995 that $\gamma(K) > 0$ (this includes the case $\gamma = \infty$) if and only if K supports a nontrivial positive measure with linear growth and finite Menger curvature [11].

Melnikov proved the “if” part of his own conjecture in the same article and the “only if” part was proved by X. Tolsa in 2003 [14].

As for harmonic functions, a theorem of Carleson [2] states that if K is a compact subset of \mathbb{R}^d ($d \geq 2$) then K is removable for harmonic functions satisfying a Lip_α condition ($0 < \alpha < 1$) if and only if $m_{d-2+\alpha}(K) = 0$.

Carleson's result fails for $\alpha = 1$: N. X. Uy [16] gave an example of a compact subset of \mathbb{R}^d that is removable for Lip_1 harmonic functions in spite of having positive $(d - 1)$ -dimensional Hausdorff measure. N. X. Uy noted that for $d = 2$ the existence of such a compact set follows from the existence of a set of positive length that is removable for bounded holomorphic functions. Uy used the above-mentioned example of Garnett in the 2-dimensional case and he generalized this example in higher dimensional Euclidean spaces.

D. Ullrich showed in 1990 [15] that K is removable for harmonic functions in the Zygmund class if and only if $m_{d-1}(K) = 0$.

When $0 < \alpha < 1$ then Lip_α coincides with the Besov space $\Lambda_\alpha^{\infty,\infty}$ (see definition below) and the Zygmund class coincides with the Besov space $\Lambda_1^{\infty,\infty}$.

All these facts indicate that Carleson-Ullrich's theorem may have generalizations in more general Besov spaces. This dissertation consists of research done in this direction. We study removable sets for harmonic functions in Besov spaces.

The main result of this dissertation is:

Consider α, p, q such that $(2 - \frac{d}{p'})_+ < \alpha < 2$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and a compact $K \subset \mathbb{R}^d$. We fix the number $\beta = (\alpha - 2)p' + d$ where $\frac{1}{p'} + \frac{1}{p} = 1$. Then:

- i)* If there exists $\gamma < \beta$ such that $m_\gamma(K) = 0$ then K is $\Lambda_{\alpha,loc}^{p,q}$ -removable;
- ii)* If there exists $\gamma > \beta$ such that $m_\gamma(K) > 0$ then K is not $\Lambda_{\alpha,loc}^{p,q}$ -removable. (See the definitions below.)

1.2 Preliminaries

In this section we state basic definitions and theorems which are instrumental in our research. We start with notations: \mathbb{R}^d denotes d -dimensional Euclidean space, \mathbb{Z} the set of all integers, \mathbb{N} the set of all natural numbers, \mathbb{R} the set of all real numbers, \mathbb{C} the set of complex numbers.

We denote by A_1 the “area” of the unit sphere in \mathbb{R}^d .

Definition 1.2.1. *By g we always denote the Green function [7] which is defined in \mathbb{R}^d as*

$$g(x) = \begin{cases} \frac{-1}{(d-2)A_1|x|^{d-2}}, & \text{if } x \neq 0, \\ -\infty, & \text{if } x = 0, \end{cases}$$

when $d > 2$,

$$g(x) = \begin{cases} \frac{-1}{2\pi} \ln |x|, & \text{if } x \neq 0, \\ -\infty, & \text{if } x = 0, \end{cases}$$

when $d = 2$ and

$$g(x) = \frac{1}{2}|x|$$

when $d = 1$.

Definition 1.2.2. *The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is*

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

For the further use, we need to make following

Remark 1.2.1. *For all $d \in \mathbb{N}$, it is true that $\hat{g}(\xi) = -\frac{1}{4\pi^2|\xi|^2}$ in $\mathbb{R}^d \setminus \{0\}$.*

Theorem 1.2.1. *Suppose that μ is a finite positive measure on \mathbb{R}^d with compact support K . Let $u = \mu * g$, where g is the Green's function (see definition 1.2.1). Then u is subharmonic, and $\Delta u = \mu$ in the sense of distributions (so that u is harmonic in $\mathbb{R}^d \setminus K$) [7].*

We denote by $m_\theta(E)$ the θ -dimensional Hausdorff measure of $E \subset \mathbb{R}^d$. We state a special case of Frostman's lemma in Euclidean spaces:

Lemma 1.2.1. *Suppose that K is a compact subset of \mathbb{R}^d and $m_\theta(K) > 0$. Then there exist a probability measure μ supported on K and a finite constant C such that*

$$\mu(B(x, r)) \leq Cr^\theta$$

for all $x \in \mathbb{R}^d$, $r > 0$.

Definition 1.2.3. Suppose μ is a measure in \mathbb{R}^d . We denote by μ_r the function

$$\mu_r(x) = \mu(B(x, r)).$$

The next theorem is known as Hardy's inequality [4].

Theorem 1.2.2. If $\gamma > 0$, $1 \leq q < \infty$, $h : (0, \infty) \rightarrow (0, \infty)$ then

$$\int_0^\infty r^{-\gamma-1} \left(\int_0^r h(t) dt \right)^q dr \leq \left(\frac{q}{r} \right)^q \int_0^\infty r^{q-\gamma-1} h(r)^q dr.$$

1.3 Basics About Besov Spaces

$\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ will be the Schwarz space of rapidly decreasing functions, with dual \mathcal{S}' , the space of tempered distributions. We will set

$$\mathcal{S}_0 = \left\{ f \in \mathcal{S} : \int_{\mathbb{R}^d} f(t) t^\gamma dt = 0, \gamma \in \mathbb{N}^d \right\},$$

so

$$\hat{\mathcal{S}}_0 = \left\{ \hat{f} : f \in \mathcal{S}_0 \right\} = \left\{ f \in \mathcal{S} : \frac{\partial^\gamma f}{\partial x^\gamma}(0) = 0, \gamma \in \mathbb{N}^d \right\}.$$

The dual of \mathcal{S}_0 is

$$\mathcal{S}'_0 = \mathcal{S}' / \mathcal{S}_0^\perp = \mathcal{S}' / \mathcal{P}$$

where \mathcal{P} is the space of all polynomials.

For each $n \in \mathbb{Z}$, the notation \mathbb{A}_n will refer to the annulus

$$\mathbb{A}_n = \{ \xi \in \mathbb{R}^d : 2^{n-1} < |\xi| < 2^{n+1} \}.$$

Choose $\psi_0 \in C^\infty(\mathbb{R}^d)$ such that $\psi_0 > 0$ on \mathbb{A}_0 while $\psi_0 = 0$ on $\mathbb{R}^d \setminus \mathbb{A}_0$. For $n \in \mathbb{Z}$ define

$$\psi_n(\xi) = \psi_0(2^{-n}\xi)$$

(so that ψ_n vanishes off \mathbb{A}_n), and now define $\Phi_n \in \mathcal{S}$ by

$$\hat{\Phi}_n(\xi) = \frac{\psi_n(\xi)}{\sum_{j=-\infty}^{\infty} \psi_j(\xi)}.$$

It follows that $\hat{\Phi}_n$ vanishes off \mathbb{A}_n , $\|\Phi_n\|_1 = \|\Phi_0\|_1$, $\Phi_n(x) = 2^{nd}\Phi_0(2^n x)$,

$$\int_{\mathbb{R}^d} x^\gamma \Phi_n(x) dx = \frac{\partial^\gamma \hat{\Phi}_n}{\partial x^\gamma}(0) = 0, \quad \gamma \in \mathbb{N}^d,$$

so $\Phi_n \in \mathcal{S}_0$, (hence $\Phi_n * f$ is defined for $f \in \mathcal{S}'_0$) and $\sum_{n=-\infty}^{\infty} \hat{\Phi}_n(\xi) = 1$ ($\xi \in \mathbb{R}^d \setminus \{0\}$).

Note that $\sum_{n=-\infty}^{\infty} \hat{\Phi}_n(0) = 0$.

Since $1 - \sum_{n=0}^{\infty} \hat{\Phi}_n$ is smooth and has compact support we can choose $\Psi \in \mathcal{S}$ such that $\hat{\Psi} = 1 - \sum_{n=0}^{\infty} \hat{\Phi}_n$; note that $\hat{\Psi}$ vanishes off $B(0, 1)$ (the ball with center at the origin and radius 1).

Now $\hat{\Psi} + \sum_{n=0}^{\infty} \hat{\Phi}_n = 1$ in all of \mathbb{R}^d , and in fact it is easy to show that

$$\phi = \Psi * \phi + \sum_{n=0}^{\infty} \Phi_n * \phi,$$

with convergence in \mathcal{S} , for all $\phi \in \mathcal{S}$. It follows that

$$f = \Psi * f + \sum_{n=0}^{\infty} \Phi_n * f$$

in \mathcal{S}' for all $f \in \mathcal{S}'$ (Statements about convergence in \mathcal{S}' and \mathcal{S}'_0 refer to the *weak** topology).

Typically when we write $f = \sum_{n=-\infty}^{\infty} f_n$ below it will be understood that we mean convergence in \mathcal{S}'_0 and that \hat{f}_n vanishes off \mathbb{A}_n .

Similarly, the notation $f = F + \sum_{n=0}^{\infty} f_n$ will be taken to imply convergence in \mathcal{S}' , that \hat{f}_n vanishes off \mathbb{A}_n , and that \hat{F} vanishes off $B(0, 1)$.

Note that f_n and F will always be infinitely differentiable, since \hat{f}_n and \hat{F} are distributions with compact support.

Definition 1.3.1. Suppose that $-\infty < \alpha < \infty$ and $p \in [1, \infty]$, $q \in [1, \infty)$. The homogeneous Besov space $\dot{\Lambda}_{\alpha}^{p,q} = \dot{\Lambda}_{\alpha}^{p,q}(\mathbb{R}^d)$ is the space of tempered distributions $f \in \mathcal{S}'_0$ such that $f = \sum_{n=-\infty}^{\infty} f_n$ where

$$\text{supp}(\hat{f}_n) \subset \mathbb{A}_n$$

and

$$\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|f_n\|_p)^q < \infty.$$

If $q = \infty$ this condition becomes $2^{n\alpha} \|f_n\|_p \leq C$ for all $n \in \mathbb{Z}$.

Definition 1.3.2. Suppose that $-\infty < \alpha < \infty$ and $1 \leq p \leq \infty$, $1 \leq q < \infty$. The inhomogeneous Besov space $\Lambda_{\alpha}^{p,q} = \Lambda_{\alpha}^{p,q}(\mathbb{R}^d)$ is the space of tempered distributions $f \in \mathcal{S}'$ such that $f = F + \sum_{n=0}^{\infty} f_n$, where

$$\text{supp}(\hat{F}) \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}, \text{supp}(\hat{f}_n) \subset \mathbb{A}_n$$

and

$$\|F\|_p + \left(\sum_{n=0}^{\infty} (2^{n\alpha} \|f_n\|_p)^q \right)^{\frac{1}{q}} < \infty.$$

If $q = \infty$ this condition becomes $F \in L^p$ and $2^{n\alpha} \|f_n\|_p \leq C$ for all $n \in \mathbb{N}$.

We will let $\mathring{\Lambda}_\alpha = \mathring{\Lambda}_\alpha^{\infty, \infty}$ and $\Lambda_\alpha = \Lambda_\alpha^{\infty, \infty}$.

Norms in $\mathring{\Lambda}_\alpha^{p, q}$ and $\Lambda_\alpha^{p, q}$ are introduced by formulas:

$$\|f\|_{\alpha, p, q}^\circ = \begin{cases} \left(\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|\Phi_n * f\|_p)^q \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \sup_{n \in \mathbb{Z}} 2^{n\alpha} \|\Phi_n * f\|_p, & \text{if } q = \infty, \end{cases}$$

and

$$\|f\|_{\alpha, p, q} = \begin{cases} \|\Psi * f\|_p + \left(\sum_{n=0}^{\infty} (2^{n\alpha} \|\Phi_n * f\|_p)^q \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \|\Psi * f\|_p + \sup_{n \in \mathbb{N}} 2^{n\alpha} \|\Phi_n * f\|_p & \text{if } q = \infty, \end{cases}$$

correspondingly.

With respect to those norms Besov spaces are Banach spaces [12].

Note: We will be considering only the case $\alpha > 0$; in this case if f is in $\Lambda_\alpha^{p, q}$ or $\mathring{\Lambda}_\alpha^{p, q}$ it follows that f is locally integrable (see Lemma 2.2.3 below).

Definition 1.3.3. *Lipschitz space (homogeneous) Lip_α° ($0 < \alpha \leq 1$) is the subspace of $C(\mathbb{R}^d)/\mathbb{C}$ with norm*

$$\|f\|_{Lip_\alpha^\circ} = \sup_{\substack{x, y \in \mathbb{R}^d \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Definition 1.3.4. *Lipschitz space (inhomogeneous) Lip_α ($0 < \alpha \leq 1$) is the subspace of $C(\mathbb{R}^d)$ with norm*

$$\|f\|_{Lip_\alpha} = \|f\|_\infty + \sup_{\substack{x, y \in \mathbb{R}^d \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Definition 1.3.5. *We say that $f \in Zygm^\circ$ (homogeneous Zygmund class) if $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is a continuous function satisfying*

$$|f(x - h) - 2f(x) + f(x + h)| \leq C|h|$$

for all $h \in \mathbb{R}^d$.

Definition 1.3.6. *We say $f \in Zygm$ (the Zygmund class) if $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is bounded, measurable and satisfies*

$$|f(x - h) - 2f(x) + f(x + h)| \leq C|h|$$

for all $h \in \mathbb{R}^d$.

The next theorem, which belongs to the folklore [12], gives relationships among Besov spaces, Lipschitz spaces and the Zygmund class:

Theorem 1.3.1. *$Lip_\alpha^\circ = \mathring{\Lambda}_\alpha$ and $Lip_\alpha = \Lambda_\alpha$ when $0 < \alpha < 1$; $Zygm^\circ = \mathring{\Lambda}_1$ and $Zygm = \Lambda_1$.*

In addition to this theorem we note well known facts: $Lip_1^\circ \subset Zygm^\circ$ ($Lip_1 \subset Zygm$) but $Lip_1^\circ \neq Zygm^\circ$ ($Lip_1 \neq Zygm$).

Definition 1.3.7. *The Poisson kernel in \mathbb{R}^d is the function \mathbf{P}_y defined by*

$$\mathbf{P}_y(x) = p_d \frac{y}{(y^2 + |x|^2)^{\frac{d+1}{2}}} \quad (x \in \mathbb{R}^d, y > 0),$$

where p_d is chosen so that $\int_{\mathbb{R}^d} \mathbf{P}_y(x) dx = 1$.

The Poisson integral of f is the function in \mathbb{R}_+^{d+1} defined by

$$\mathbf{P}[f](x, y) = f * \mathbf{P}_y(x)$$

where \mathbb{R}_+^{d+1} is the upper half space of \mathbb{R}^{d+1} :

$$\mathbb{R}_+^{d+1} = \{(x, y) : x \in \mathbb{R}^d, y > 0\}.$$

Definition 1.3.8. *If $u(x, y)$ is a function in \mathbb{R}_+^{d+1} and $y > 0$ then for $1 \leq p < \infty$ we*

have

$$\mathbf{m}_{y,p}(u) = \left(\int_{\mathbb{R}^d} |u(x, y)|^p dx \right)^{\frac{1}{p}},$$

and

$$\mathbf{m}_{y,\infty}(u) = \sup_{x \in \mathbb{R}} |u(x, y)|$$

for $p = \infty$.

We define the finite difference $\Delta_h f$ by

$$\Delta_h f(x) = f(x + h) - f(x),$$

and we define higher-order difference operators by induction:

$$\Delta_h^1 f = \Delta_h f, \quad \Delta_h^{k+1} f = \Delta_h(\Delta_h^k f).$$

Theorem 1.3.2. [12] *Suppose that $\alpha > 0$, $p \in [1, \infty]$, $q \in [1, \infty)$, $k \in \mathbb{Z}$, and $k > \alpha$.*

Let $u = \mathbf{P}[f]$. The following are equivalent:

- (i) $f \in \dot{\Lambda}_\alpha^{p,q}$,
- (ii) $\left(\int_{\mathbb{R}^d} \left(\frac{\|\Delta_h^k f\|_p}{|h|^\alpha} \right)^q \frac{dh}{|h|^d} \right)^{\frac{1}{q}} < \infty$,
- (iii) $\left(\int_0^\infty \left(\frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial y^k} \right)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty$,
- (iv) *If β is any multi-index with $|\beta| = k$ then $\left(\int_0^\infty \left(\frac{y^k \mathbf{m}_{y,p}(\mathbf{D}^\beta u)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty$,*
- (v) $\left(\int_0^\infty \left(\frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial x_j^k} \right)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty$ *for $j = 1, \dots, d$.*

When $q = \infty$ then (ii), (iii), (iv), and (v) should be interpreted as:

- (ii') $\sup_{h \neq 0} \frac{\|\Delta_h^k f\|_p}{|h|^\alpha} < \infty$,
- (iii') $\sup_{y > 0} \frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial y^k} \right)}{y^\alpha} < \infty$,
- (iv') $\sup_{y > 0} \frac{y^k \mathbf{m}_{y,p}(\mathbf{D}^\beta u)}{y^\alpha} < \infty$,
- (v') $\sup_{y > 0} \frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial x_j^k} \right)}{y^\alpha} < \infty$ *for $j = 1, \dots, d$.*

Theorem 1.3.3. [12] *Suppose that $\alpha > 0$, $p \in [1, \infty]$, $q \in [1, \infty)$, $k \in \mathbb{Z}$, and $k > \alpha$.*

Let $u = \mathbf{P}[f]$. The following are equivalent:

- (i) $f \in \Lambda_\alpha^{p,q}$,
- (ii) $f \in L^p$ and $\left(\int_{|h| \leq 1} \left(\frac{\|\Delta_h^k f\|_p}{|h|^\alpha} \right)^q \frac{dh}{|h|^d} \right)^{\frac{1}{q}} < \infty$,

$$(iii) \ f \in L^p \text{ and } \left(\int_0^1 \left(\frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial y^k} \right)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty,$$

$$(iv) \ f \in L^p \text{ and if } \beta \text{ is any multi-index with } |\beta| = k \text{ then } \left(\int_0^1 \left(\frac{y^k \mathbf{m}_{y,p} (\mathbf{D}^\beta u)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty,$$

$$(v) \ f \in L^p \text{ and } \left(\int_0^1 \left(\frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial x_j^k} \right)}{y^\alpha} \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty \text{ for } j = 1, \dots, d.$$

When $q = \infty$ then (ii), (iii), (iv), and (v) should be interpreted as:

$$(ii') \ f \in L^p \text{ and } \sup_{h \neq 0} \frac{\|\Delta_h^k f\|_p}{|h|^\alpha} < \infty,$$

$$(iii') \ f \in L^p \text{ and } \sup_{y > 0} \frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial y^k} \right)}{y^\alpha} < \infty,$$

$$(iv') \ f \in L^p \text{ and } \sup_{y > 0} \frac{y^k \mathbf{m}_{y,p} (\mathbf{D}^\beta u)}{y^\alpha} < \infty,$$

$$(v') \ f \in L^p \text{ and } \sup_{y > 0} \frac{y^k \mathbf{m}_{y,p} \left(\frac{\partial^k u}{\partial x_j^k} \right)}{y^\alpha} < \infty \text{ for } j = 1, \dots, d.$$

The following theorem is known as the Besov Embedding Theorem [12]:

Theorem 1.3.4. *If $1 \leq p \leq r \leq \infty$ and $\alpha = \beta + \frac{1}{p} - \frac{1}{r}$ then*

$$\Lambda_\alpha^{p,q} \subset \Lambda_\beta^{r,q}$$

and

$$\mathring{\Lambda}_\alpha^{p,q} \subset \mathring{\Lambda}_\beta^{r,q}$$

for all $q \in [1, \infty]$.

The next two lemmas will be very useful for us in the future [12].

Lemma 1.3.1. *$f \in \mathring{\Lambda}_\alpha^{p,q}$ if and only if $D^\gamma(f) \in \mathring{\Lambda}_{\alpha-|\gamma|}^{p,q}$ where $-\infty < \alpha < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\gamma \in \mathbb{N}^d$.*

Lemma 1.3.2. *If $f \in \Lambda_{\alpha}^{p,q}$ then $D^{\gamma}(f) \in \Lambda_{\alpha-|\gamma|}^{p,q}$ where $-\infty < \alpha < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\gamma \in \mathbb{N}^d$.*

Before we state the theorem about duals of the Besov spaces [12], we need to introduce the next notations.

Suppose $h \in \Lambda_{-\alpha}^{p',q'}$ and $f \in \Lambda_{\alpha}^{p,q}$ where as always below p' and q' are the conjugate exponents to p and q , respectively. Note that

$$\int_{\mathbb{R}^d} (\Phi_n * f)(x)(\Phi_m * h)(x) dx = 0$$

when $|n-m| > 1$, because the (distribution) Fourier transforms have disjoint support.

Similarly

$$\int_{\mathbb{R}^d} (\Psi * f)(x)(\Phi_n * h)(x) dx = \int_{\mathbb{R}^d} (\Phi_n * f)(x)(\Psi * h)(x) dx = 0$$

for $n \geq 1$. We define a linear functional $\lambda_h : \Lambda_{\alpha}^{p,q} \rightarrow \mathbb{C}$ by a formula

$$\begin{aligned} \lambda_h(f) = & \int_{\mathbb{R}^d} (\Psi * f)(x)(\Psi * h)(x) dx + \int_{\mathbb{R}^d} (\Phi_0 * f)(x)(\Psi * h)(x) dx + \int_{\mathbb{R}^d} (\Psi * f)(x)(\Phi_0 * h)(x) dx \\ & + \int_{\mathbb{R}^d} (\Phi_0 * f)(x)(\Phi_0 * h)(x) dx + \sum_{j=-1}^1 \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} (\Phi_n * f)(x)(\Phi_{n+j} * h)(x) dx. \end{aligned}$$

The norm closure of the Schwarz space in $\Lambda_{\alpha}^{p,q}$ we denote by $\Lambda_{\alpha,0}^{p,q}$ and observe that $\Lambda_{\alpha}^{p,q} = \Lambda_{\alpha,0}^{p,q}$ except when $q = \infty$.

Theorem 1.3.5. *Suppose $\alpha \in \mathbb{R}$, $1 \leq q \leq \infty$, $1 \leq p \leq \infty$. If $\lambda \in (\Lambda_{\alpha,0}^{p,q})^*$ then there exists $h \in \Lambda_{-\alpha}^{p',q'}$ with $\lambda = \lambda_h$.*

An atomic decomposition of distributions from Besov spaces, introduced by M. Frazier and B. Jawerth [5], plays a very important role in our research.

Definition 1.3.9. An (α, p) -atom $a(x)$, $(-\infty < \alpha < \infty, 0 < p \leq \infty)$ is a function satisfying, for some cube $Q \subseteq \mathbb{R}^d$, the following conditions:

$$(i) \text{ supp } (a) \subseteq 3Q,$$

$$(ii) |D^\gamma a(x)| \leq |Q|^{\frac{\alpha}{d} - \frac{1}{p} - \frac{|\gamma|}{d}} \text{ if } |\gamma| \leq K, \text{ and } x \in \mathbb{R}^d$$

$$(iii) \int_{\mathbb{R}^d} x^\gamma a(x) dx = 0 \text{ if } |\gamma| \leq N,$$

where $K \geq ([\alpha] + 1)_+$ and $N \geq \max \left([d \left(\frac{1}{p} - 1 \right)_+ - \alpha], -1 \right)$ are fixed integers.

Here $x_+ = \max(x, 0)$, $[x]$ is the greatest integer in x , and $3Q$ is the cube in \mathbb{R}^d concentric with Q but with side length three times of Q . If $N = -1$ then $a(x)$ is not required to have any vanishing moments.

We note that if $\alpha > 0$ and $p \geq 1$ then we may take $N = -1$ so that condition (iii) may be ignored.

We write a_Q for an atom satisfying the definition 1.3.9 for a given cube Q , and adopt the convention that whenever Q appears as a summation index, the sum runs only over dyadic cubes. We also need for future use the following notations: Fix $\psi \in \mathcal{S}$ satisfying $\text{supp } (\hat{\psi}) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq \pi\}$, $\int_{\mathbb{R}^d} x^\eta \psi(x) dx = 0$ if $|\eta| \leq N$, and $\hat{\psi}(\xi) \geq c > 0$ if $1/2 \leq |\xi| \leq 2$ (N is the fixed integer above). For each $\nu \in \mathbb{Z}$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, set

$$Q_{\nu k} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : k_i 2^{-\nu} \leq x_i < (k_i + 1) 2^{-\nu}, i = 1, \dots, d\} \quad (1.1)$$

and define

$$\psi_Q(x) = |Q|^{\frac{\alpha}{d}-\frac{1}{p}} \psi(2^\nu x - k) \quad \text{if } Q = Q_{\nu k}. \quad (1.2)$$

Definition 1.3.10. We call a function m an (α, p) -molecule if there exist $\mu \in \mathbb{Z}$ and a point $x_0 \in \mathbb{R}^d$ such that

$$|D^\eta m(x)| \leq 2^{\mu(\frac{d}{p}-\alpha+|\eta|)} (1 + 2^\mu |x - x_0|)^{-M-|\eta|} \quad \text{if } |\eta| \leq K$$

and

$$\int_{\mathbb{R}^d} x^\eta m(x) dx = 0 \quad \text{if } |\eta| \leq N,$$

where $M \geq N + 10d$ and K, N are as above.

Suppose $\Psi \in \mathcal{S}$ satisfies: $\text{supp } (\hat{\Psi}) \subseteq \{\xi : |\xi| \leq \pi\}$ and $\hat{\Psi}(\xi) \geq C > 0$ if $|\xi| \leq 1$.

Then the following decomposition result is true [5].

Theorem 1.3.6. Let $-\infty < \alpha < \infty$, $0 < p, q \leq \infty$.

a) Each $f \in \Lambda_\alpha^{p,q}$ can be decomposed as follows:

$$i) \quad f(\cdot) = \sum_{k \in \mathbb{Z}^d} s_k \Psi(\cdot - k) + \sum_{\nu=0}^{\infty} \sum_{l(Q)=2^{-\nu}} s_Q \psi_Q(\cdot),$$

where the ψ_Q 's are defined by (1.2), or

$$ii) \quad f = \sum_{k \in \mathbb{Z}^d} s_k b_k + \sum_{\nu=0}^{\infty} \sum_{l(Q)=2^{-\nu}} s_Q a_Q,$$

where the a_Q 's are (α, p) -atoms, and the b_k 's satisfy $\text{supp } b_k \subseteq 3Q_{0k}$ and $|D^\eta b_k(x)| \leq 1$

if $|\eta| \leq K$. In both cases the convergence is in \mathcal{S}' , and

$$\left(\sum_{k \in \mathbb{Z}^d} |s_k|^p \right)^{\frac{1}{p}} + \left(\sum_{\nu=0}^{\infty} \left(\sum_{l(Q)=2^{-\nu}} |s_Q|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \|f\|_{\alpha, p, q},$$

with C independent of f .

b) Conversely, suppose $f = \sum_{k \in \mathbb{Z}^d} s_k m_k + \sum_{\nu=0}^{\infty} \sum_{l(Q)=2^{-\nu}} s_Q m_Q$, where each m_Q is an (α, p) -molecule concentrated on Q , and each m_K satisfies $|D^\eta m_k(x)| \leq (1 + |x - k|)^{-M-|\eta|}$ if $|\eta| \leq K$, for some sufficiently large M .

Then

$$\|f\|_{\alpha, p, q} \leq C \left(\sum_{k \in \mathbb{Z}^d} |s_k|^p \right)^{\frac{1}{p}} + C \left(\sum_{\nu=0}^{\infty} \left(\sum_{l(Q)=2^{-\nu}} |s_Q|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

The next theorems on Besov spaces are crucial for our purpose to characterize removable sets for harmonic functions in Besov spaces. Theorem 1.3.7 is an unpublished result of D. Ullrich and by courtesy of the author we are providing the proofs in this thesis.

Definition 1.3.11. Suppose f is a locally integrable function. For $r > 0$ we define:

$$S_r f(x) = f(x) - \int_S f(x + r\xi) d\sigma(\xi)$$

where σ is the rotation-invariant probability measure on S , the sphere of radius 1.

Lemma 1.3.3. Suppose that $k > \frac{d}{2}$ is an integer. There exists $C > 0$ such that for any $f \in C_c^k(\mathbb{R}^d)$ we have

$$\|\hat{f}\|_1 \leq C \|f\|_2^{1-\frac{d}{2k}} \left(\sum_{i=1}^d \left\| \frac{\partial^k f}{\partial x_i^k} \right\|_2 \right)^{\frac{d}{2k}}.$$

Proof. We observe that we are using Hölder's inequality and Plancherel's theorem.

$$\begin{aligned}
\int_{\mathbb{R}^d} |\hat{f}(x)| dx &= \int_{B(0,A)} |\hat{f}(x)| dx + \int_{\mathbb{R}^d \setminus B(0,A)} |\hat{f}(x)| dx \\
&= \int_{B(0,A)} |\hat{f}(x)| dx + \int_{\mathbb{R}^d \setminus B(0,A)} \frac{\left(\sum_{i=1}^d |x_i|^k \right) |\hat{f}(x)|}{\sum_{i=1}^d |x_i|^k} dx \\
&\leq C_1 A^{\frac{d}{2}} \|f\|_2 + C_2 \left(\sum_{i=1}^d \left\| \frac{\partial^k f}{\partial x_i^k} \right\|_2 \right) A^{-k+\frac{d}{2}}.
\end{aligned}$$

Let

$$A = \left(\frac{C_2 \left(\sum_{i=1}^d \left\| \frac{\partial^k f}{\partial x_i^k} \right\|_2 \right)}{C_1 \|f\|_2} \right)^{\frac{1}{k}}.$$

□

Theorem 1.3.7. Suppose that $0 < \alpha < 2$. Then $f \in \dot{\Lambda}_\alpha^{p,q}$ if and only if

$$\int_0^\infty \left(\frac{\|S_r f\|_p}{r^\alpha} \right)^q \frac{dr}{r} < \infty$$

which for $q = \infty$ should be taken to mean that $\frac{\|S_r f\|_p}{r^\alpha}$ is bounded for $0 < r < \infty$.

Proof. We consider only the case $1 \leq q < \infty$; the case $q = \infty$ is very similar.

Assume that $f \in \dot{\Lambda}_\alpha^{p,q}$. Note that

$$\begin{aligned}
S_r f(x) &= f(x) - \int_S f(x + r\xi) d\sigma(\xi) \\
&= f(x) - \int_S \frac{f(x + r\xi) + f(x - r\xi)}{2} d\sigma(\xi)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \int_S (f(x+r\xi) - 2f(x) + f(x-r\xi)) \, d\sigma(\xi) \\
&= -\frac{1}{2} \int_S \Delta_{r\xi}^2 f(x) \, d\sigma(\xi).
\end{aligned}$$

Therefore

$$\|S_r f\|_p \leq C \int_S \|\Delta_{rh}^2 f\|_p \, d\sigma(h),$$

so that Hölder's inequality gives

$$\|S_r f\|_p^q \leq C \int_S \|\Delta_{rh}^2 f\|_p^q \, d\sigma(h),$$

and an integration in polar coordinates shows that:

$$\begin{aligned}
\int_0^\infty \left(\frac{\|S_r f\|_p}{r^\alpha} \right)^q \frac{dr}{r} &\leq C \int_0^\infty \frac{\int_S \|\Delta_{rh}^2 f\|_p^q \, d\sigma(h)}{r^{\alpha q}} \frac{dr}{r} \\
&= C \int_0^\infty \int_S \left(\frac{\|\Delta_{rh}^2 f\|_p}{|rh|^\alpha} \right)^q \frac{1}{|rh|^d} \, d\sigma(h) r^{d-1} \, dr \\
&= C \int_{\mathbb{R}^d} \left(\frac{\|\Delta_h^2 f\|_p}{|h|^\alpha} \right)^q \frac{dh}{|h|^d} < \infty.
\end{aligned}$$

To finish the proof of the first part we use theorem 1.3.2.

To prove the second part it is sufficient to show that there exists $\delta > 0$ and $C > 0$ such that for all integers n we have

$$\|\Phi_n * f\|_p \leq C \|S_r f\|_p, \quad (2^{-n}\delta \leq r \leq 2^{-n+1}\delta).$$

We will prove this in case $n = 0$; the general case follows by dilation.

From the definition of $\|f\|_{p,q,\alpha}^\circ$ it follows that

$$\|f\|_{p,q,\alpha}^\circ \leq C \left(\int_0^\infty \left(\frac{\|S_r f\|_p}{r^\alpha} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}},$$

which is enough for the proof.

We have:

$$\widehat{S_r f} = \nu_r \hat{f},$$

where

$$\nu_r(\xi) = 1 - \int_S e^{irh \cdot \xi} d\sigma(h) = 1 - J(r|\xi|),$$

where in turn

$$J(r) = \int_S e^{ir\zeta_1} d\sigma(\zeta).$$

It is clear that J is smooth, and well-known that $J(r) \rightarrow 0$ as $r \rightarrow \infty$. (We are assuming that $d > 1$ here; in the case $d = 1$ there is nothing to prove, because $S_r f = -\frac{\Delta_r^2 f}{2}$). So there exists $\delta > 0$ such that $|J(r)| < \frac{1}{2}$ ($r > \frac{\delta}{2}$).

This implies that $|\nu_r(\xi)| \geq \frac{1}{2}$ ($\delta \leq r \leq 2\delta, \xi \in \mathbb{A}_0$).

Now let N be an integer larger than $\frac{d}{2}$. Since J is smooth all its partial derivatives are locally bounded; it follows from the quotient rule and the fact that $|\nu_r| \geq \frac{1}{2}$ that

$$\left| \frac{\partial^\beta \widehat{\Phi}_0(\xi)}{\partial \xi^\beta \nu_r(\xi)} \right| \leq C \quad (\delta \leq r \leq 2\delta, \xi \in \mathbb{A}_0)$$

whenever β is a multi-index with $|\beta| \leq N$. Since all our functions are supported in \mathbb{A}_0 it follows from Lemma 1.3.3 that there exists $\phi_r \in L^1$ such that

$$\hat{\phi}_r = \frac{\hat{\Phi}_0}{\nu_r}$$

and

$$\|\phi_r\|_1 \leq C \quad (\delta \leq r \leq 2\delta).$$

This shows that

$$f * \Phi_0 = \phi_r * (S_r f),$$

so that

$$\|\Phi_0 * f\|_p \leq \|\phi_r\|_1 \|S_r f\|_p \leq C \|S_r f\|_p \quad (\delta \leq r \leq 2\delta).$$

That is exactly the inequality we need for $n = 0$. □

Theorem 1.3.7 plus a little argument gives the corresponding result for $\Lambda_{p,q}^\alpha$:

Theorem 1.3.8. *Suppose that $0 < \alpha < 2$. Then $f \in \Lambda_{\alpha}^{p,q}$ if and only if $f \in L^p$ and*

$$\int_0^1 \left(\frac{\|S_r f\|_p}{r^\alpha} \right)^q \frac{dr}{r} < \infty$$

which for $q = \infty$ should be taken to mean that $\frac{\|S_r f\|_p}{r^\alpha}$ is bounded for $0 < r \leq 1$.

Lemma 1.3.4. *If $d > 2$ then*

$$S_r(g)(x) = \begin{cases} \frac{-1}{A_1(d-2)}(r^{2-d} - |x|^{2-d}), & \text{if } |x| \leq r, \\ 0, & \text{otherwise,} \end{cases}$$

where A_1 is the “area” of the unit sphere in \mathbb{R}^d .

When $d = 2$ we have

$$S_r(g)(x) = \begin{cases} \frac{-1}{2\pi}(\ln |r| - \ln |x|), & \text{if } |x| \leq r, \\ 0, & \text{otherwise,} \end{cases}$$

and when $d = 1$ we have

$$S_r(g)(x) = \begin{cases} \frac{1}{2}(|x| - r), & \text{if } |x| \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We just give a proof when $d > 2$; $d = 2$ and $d = 1$ are similar. The fact that $S_r(g)(x) = 0$ for $|x| > r$ is clear since g is harmonic in $\mathbb{R}^d \setminus \{0\}$. Note that $S_r(g) = g - g * \sigma_r$ where σ_r is the rotation-invariant probability measure on rS , the sphere of radius r . It is clear that $\sigma_r * g$ is radial since σ_r and g are both radial (rotation invariant).

Therefore, in $B(0, r)$ the function $\sigma_r * g$ is radial and harmonic, so it must be constant since it satisfies the mean value property; for $|x| < r$ we have

$$\sigma_r * g(x) = \sigma_r * g(0) = \int_S g(r\xi) d\sigma(\xi) = \frac{-r^{2-d}}{A_1(d-2)}.$$

Thus $S_r(g)(x) = g(x) - \sigma_r * g(x) = \frac{-1}{A_1(d-2)}(r^{2-d} - |x|^{2-d})$. \square

Lemma 1.3.5. $S_r(g)(x) = \frac{1}{A_1} \int_0^r t^{1-d} \mathbf{1}_{B(0,t)}(x) dt$ where

$$\mathbf{1}_{B(0,t)}(x) = \begin{cases} 1, & \text{if } |x| < t, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the case when $d > 2$. The cases $d = 1$ and $d = 2$ are very similar.

If $|x| > r$ it is clear that $\int_0^r t^{1-d} \mathbf{1}_{B(0,t)}(x) dt = 0$. If $|x| < r$ then $\frac{1}{A_1} \int_0^r t^{1-d} \mathbf{1}_{B(0,t)}(x) dt = \frac{1}{A_1} \int_{|x|}^r t^{1-d} dt = \frac{-1}{A_1(d-2)} (r^{2-d} - |x|^{2-d})$. \square

2 Conditions of Non-Removability

2.1 Local Besov Spaces

Definition 2.1.1. We say that a distribution f belongs to local Besov space and write

$f \in \Lambda_{\alpha,loc}^{p,q}$ if for all $\varphi \in C_c^\infty$ we have $\varphi f \in \Lambda_\alpha^{p,q}$.

The next proposition gives an equivalent definition of local Besov spaces:

Theorem 2.1.1. Suppose $0 < \alpha < 2$, and $1 \leq p \leq \infty$. Then $f \in \Lambda_{\alpha,loc}^{p,\infty}$ if and only if $f \in L_{loc}^p(\mathbb{R}^d)$ and for any compact $K \subset \mathbb{R}^d$ there exists $C > 0$ such that

$$\left(\int_K |\Delta_h^2 f(x)|^p dx \right)^{\frac{1}{p}} \leq C|h|^\alpha \quad (0 < |h| \leq 1).$$

Proof. We adopt the common convention that the letter C denotes a constant, the value of which may vary from line to line. The “only if” part is trivial so we focus on the “if” part. Assume $f \in L_{loc}^p(\mathbb{R}^d)$ and for any compact $K \subset \mathbb{R}^d$ there exists $C > 0$ such that

$$\left(\int_K |\Delta_h^2 f(x)|^p dx \right)^{\frac{1}{p}} \leq C|h|^\alpha \quad (0 < |h| \leq 1).$$

Suppose $\phi \in C_c^\infty(\mathbb{R}^d)$; let $K = \text{supp}(\phi)$, $K_1 = \bigcup_{x \in K} \overline{B(x, 2)}$. We know that

$$\Delta_h(\phi f) = (\Delta_h \phi)(\tau_h f) + \phi \Delta_h f.$$

Applying that formula twice shows that

$$\Delta_h^2(\phi f) = (\Delta_h^2 \phi)(\tau_{2h} f) + 2(\Delta_h \phi)(\tau_h \Delta_h f) + \phi \Delta_h^2 f.$$

Since ϕ has compact support and the function f is locally integrable everywhere it would be enough to give an L^p -norm estimate of

$$(\Delta_h^2 \phi)f + 2(\Delta_h \phi)(\Delta_h f) + \phi \Delta_h^2 f$$

in order to have an L^p -norm estimate of $\Delta_h^2(\phi f)$. By Theorem 1.3.3 (ii'), it is enough to show that $\|\Delta_h^2(\phi f)\|_p \leq C|h|^\alpha$.

If $|h| > 1$ then

$$\|\Delta_h^2(\phi f)\|_p \leq 4\|\phi f\|_p = C \leq C|h|^\alpha.$$

So we only need to consider the case $0 < |h| \leq 1$.

In this case we have

$$\|(\Delta_h^2 \phi)f\|_p \leq \|\Delta_h^2 \phi\|_\infty \left(\int_{K_1} |f|^p \right)^{\frac{1}{p}} \leq C|h|^2 \leq C|h|^\alpha,$$

since $\phi \in C_c^\infty(\mathbb{R}^d)$ and $0 < \alpha < 2$. Also, it is easy to see that

$$\|\phi \Delta_h^2 f\|_p \leq C \left(\int_K |\Delta_h^2 f|^p \right)^{\frac{1}{p}} \leq C|h|^\alpha.$$

Thus we have the $(\Delta_h \phi)(\Delta_h f)$ term left to evaluate. We observe that $\Delta_h^2 f = (\tau_{2h} f - \tau_h f) - (\tau_h f - f)$ so

$$\begin{aligned}\Delta_{2h} f &= (\tau_{2h} f - \tau_h f) + (\tau_h f - f) \\ &= \Delta_h^2 f + 2(\tau_h f - f) \\ &= \Delta_h^2 f + 2\Delta_h f,\end{aligned}$$

or $\Delta_h f - \frac{1}{2}\Delta_{2h} f = -\frac{1}{2}\Delta_h^2 f$. Now assume that $2^{-N} \leq |h| < 2^{-N+1}$. Then

$$\begin{aligned}\|\Delta_h f\|_{L^p(K_1)} &\leq \|\Delta_h f - \frac{1}{2}\Delta_{2h} f\|_{L^p(K_1)} \\ &\quad + \|\frac{1}{2}\Delta_{2h} f - \frac{1}{4}\Delta_{4h} f\|_{L^p(K_1)} \\ &\quad + \cdots \\ &\quad + \|2^{-N+1}\Delta_{2^{N-1}h} f - 2^{-N}\Delta_{2^N h} f\|_{L^p(K_1)} \\ &\quad + 2^{-N}\|\Delta_{2^N h} f\|_{L^p(K_1)} \\ &= \frac{1}{2}\|\Delta_h^2 f\|_{L^p(K_1)} + \cdots + 2^{-N}\|\Delta_{2^{N-1}h}^2 f\|_{L^p(K_1)} + 2^{-N}\|\Delta_{2^N h} f\|_{L^p(K_1)}.\end{aligned}$$

Since $2^N |h| < 2$, we can easily estimate the last term:

$$2^{-N}\|\Delta_{2^N h} f\|_{L^p(K_1)} \leq 2|h|\|f\|_{L^p(K_2)} = C|h|$$

where $K_2 = \bigcup_{x \in K_1} \overline{B(x, 1)}$.

Now we estimate the other terms:

$$\begin{aligned}
& \frac{1}{2} \|\Delta_h^2 f\|_{L^p(K_1)} + \cdots + 2^{-N} \|\Delta_{2^{N-1}h}^2 f\|_{L^p(K_1)} \\
& \leq C \left(\frac{1}{2} |h|^\alpha + \frac{1}{4} |2h|^\alpha + \cdots + 2^{-N} |2^{N-1}h|^\alpha \right) \\
& = C |h|^\alpha (1 + 2^{\alpha-1} + 2^{2(\alpha-1)} + \cdots + 2^{(N-1)(\alpha-1)}) \\
& \leq \begin{cases} C |h|^\alpha, & \text{if } 0 < \alpha < 1, \\ CN |h|^\alpha \leq C |h| \log \left(\frac{1}{|h|} \right), & \text{if } \alpha = 1, \\ C |h|^\alpha 2^{N(\alpha-1)} \leq C |h| & \text{if } \alpha > 1. \end{cases}
\end{aligned}$$

Adding in the last term shows that

$$\|\Delta_h f\|_{L^p(K_1)} = \begin{cases} C |h|^\alpha + |h|, & \text{if } 0 < \alpha < 1, \\ CN |h|^\alpha \leq C |h| \left(1 + \log \left(\frac{1}{|h|} \right) \right), & \text{if } \alpha = 1, \\ C |h|^\alpha 2^{N(\alpha-1)} \leq C |h| & \text{if } \alpha > 1. \end{cases}$$

Since $\text{supp}(\Delta_h \phi) \subset K_1$, $\|\Delta_h \phi\|_\infty \leq C |h|$, $0 < \alpha < 2$, and $|h| \leq 1$, it follows that

$$\begin{aligned}
& \|(\Delta_h \phi)(\Delta_h f)\|_p \leq C |h| \|\Delta_h f\|_{L^p(K_1)} \\
& \leq |h| \begin{cases} C |h|^\alpha + |h|, & \text{if } 0 < \alpha < 1, \\ C |h| \left(1 + \log \left(\frac{1}{|h|} \right) \right), & \text{if } \alpha = 1, \\ C |h| & \text{if } \alpha > 1. \end{cases} \\
& \leq C |h|^\alpha.
\end{aligned}$$

□

Now we give a more general definition of local Besov spaces:

Definition 2.1.2. Suppose Ω is an open set in \mathbb{R}^d . We say that a distribution f in Ω belongs to local Besov space with respect to Ω and write $f \in \Lambda_{\alpha,loc}^{p,q}(\Omega)$ if for all $\varphi \in C_c^\infty(\Omega)$ we have $\varphi f \in \Lambda_\alpha^{p,q}$.

It is easy to see that an analog of Theorem 2.1.1 is true:

Theorem 2.1.2. Suppose $0 < \alpha < 2$, and $1 \leq p \leq \infty$. Then $f \in \Lambda_{\alpha,loc}^{p,\infty}(\Omega)$ if and only if $f \in L_{loc}^p(\Omega)$ and for any compact $K \subset \Omega$ there exists $C > 0$ and $\delta > 0$ such that

$$\left(\int_K |\Delta_h^2 f(x)|^p dx \right)^{\frac{1}{p}} \leq C|h|^\alpha, \quad (0 < |h| \leq \delta).$$

We state some propositions which give relationships between local and global Besov spaces.

Proposition 2.1.1. For all $0 < \alpha < 2$, $1 \leq q \leq \infty$ and $1 \leq p \leq \infty$ we have

$$\mathring{\Lambda}_\alpha^{p,q} \subseteq \Lambda_{\alpha,loc}^{p,q}.$$

Proof. Say $f \in \mathring{\Lambda}_\alpha^{p,q}$, $f = \sum_{n=-\infty}^{\infty} f_n$. Since $\left(\sum_{n=-\infty}^{-1} f_n \right)^\wedge$ has compact support, $\sum_{n=-\infty}^{-1} f_n \in C^\infty(\mathbb{R}^d) \subset L_{loc}^p$. Also $C^\infty(\mathbb{R}^d) \subset \Lambda_{\alpha,loc}^{p,q}(\mathbb{R}^d)$ (this follows from the definition of local Besov spaces since $C_c^\infty(\mathbb{R}^d) \subset \Lambda_\alpha^{p,q}(\mathbb{R}^d)$). On the other hand because $\alpha > 0$ we have $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Since $\sum_{n=0}^{\infty} f_n \in \Lambda_\alpha^{p,q}$, it follows from Proposition 3.1.1 below that $\sum_{n=0}^{\infty} f_n \in \Lambda_{\alpha,loc}^{p,q}$. □

Proposition 2.1.2. *If $\alpha_1 < \alpha_2$ than*

$$\Lambda_{\alpha_2, loc}^{p,q}(\Omega) \subset \Lambda_{\alpha_1, loc}^{p,q}(\Omega)$$

for all $1 \leq q \leq \infty$, $1 \leq p \leq \infty$ and any open set $\Omega \subset \mathbb{R}^d$.

Proof. The proof easily follows from the fact that $\Lambda_{\alpha_2}^{p,q} \subset \Lambda_{\alpha_1}^{p,q}$ when $\alpha_1 < \alpha_2$ [11]. \square

Proposition 2.1.3. *Say Ω is an open subset of \mathbb{R}^d and K a compact subset of Ω .*

Then K is $\Lambda_{\alpha, loc}^{p,q}(\Omega)$ removable if and only if it is $\Lambda_{\alpha, loc}^{p,q}(\mathbb{R}^d)$ -removable.

Proof. For the non-trivial direction, say K is not $\Lambda_{\alpha, loc}^{p,q}(\Omega)$ -removable. Let $u \in \Lambda_{\alpha, loc}^{p,q}(\Omega)$, u harmonic in $\Omega \setminus K$, u not harmonic in Ω . Then $\Delta u \neq 0$, $\text{supp}(\Delta u) \subseteq K$ (where Δu is the distribution Laplacian of u). Since Δu has compact support and the Green function g belongs to L_{loc}^1 , there exists well defined $v = (\Delta u) * g$. Then v is harmonic in $\mathbb{R}^d \setminus K$ but not in \mathbb{R}^d , because $\Delta v = \Delta u$. To finish the proof we only need to show $v \in \Lambda_{\alpha, loc}^{p,q}(\mathbb{R}^d)$.

Fix $\varphi_0 \in C_c^\infty(\Omega)$, $\varphi_0 = 1$ on a neighborhood of K . Suppose $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then $\varphi v = \varphi \varphi_0 v + \varphi(1 - \varphi_0)v$ but $\varphi(1 - \varphi_0)v \in C_c^\infty(\mathbb{R}^d) \subset \Lambda_\alpha^{p,q}(\mathbb{R}^d)$ and $\Delta(u - v) = 0$ in Ω so $u - v \in C^\infty(\Omega)$. Hence $\varphi \varphi_0(v - u) \in C_c^\infty(\mathbb{R}^d) \subset \Lambda_\alpha^{p,q}(\mathbb{R}^d)$. On the other hand, the fact that $u \in \Lambda_{\alpha, loc}^{p,q}(\Omega)$ implies that $\varphi \varphi_0 u \in \Lambda_\alpha^{p,q}(\mathbb{R}^d)$ since $\varphi \varphi_0 \in C_c^\infty(\Omega)$. Thus

$$\varphi v = \varphi \varphi_0 v + \varphi(1 - \varphi_0)v = \varphi \varphi_0 u + \varphi \varphi_0(v - u) + \varphi(1 - \varphi_0)v \in \Lambda_\alpha^{p,q}(\mathbb{R}^d),$$

which implies that $v \in \Lambda_{\alpha, loc}^{p,q}(\mathbb{R}^d)$. \square

The previous propositions tell us that distributions from homogeneous and inhomogeneous Besov spaces are similar locally. Because of this, we will study removable compact sets for harmonic functions in local Besov spaces.

2.2 Borderline Cases

Before we move towards the general case $0 < \alpha < 2$, we are going to discuss borderline cases.

Lemma 2.2.1. *If μ is a tempered distribution and $\hat{\mu}$ has compact support then $\mu \in C^\infty(\mathbb{R}^d)$ [4].*

Corollary 2.2.1. *If $\gamma > 0$ then $\Lambda_\gamma \subseteq C(\mathbb{R}^d)$, $\dot{\Lambda}_\gamma \subseteq C(\mathbb{R}^d)$.*

Proof. If $f \in \dot{\Lambda}_\gamma$ then $f = \sum_{n=-\infty}^{\infty} f_n$, but the previous lemma implies that $\sum_{n=-\infty}^{-1} f_n \in C^\infty(\mathbb{R}^d)$, $f_n \in C^\infty(\mathbb{R}^d)$ ($n \in \mathbb{N}$). On the other hand $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$ since $\gamma > 0$, which implies the second part of this corollary. The proof of the first part is very similar. □

Corollary 2.2.2. *If $\gamma > 2$ then $\Lambda_\gamma \subseteq C^2(\mathbb{R}^d)$, $\dot{\Lambda}_\alpha \subseteq C^2(\mathbb{R}^d)$.*

Proof. The proof easily follows from Lemma 1.3.1, Lemma 1.3.2 and the previous corollary. □

Lemma 2.2.2. *Suppose $\alpha > 2 + \frac{d}{p}$ and $1 \leq p \leq \infty$. Then $\dot{\Lambda}_\alpha^{p,\infty} \subseteq C^2(\mathbb{R}^d)$.*

Proof. We know that $\dot{\Lambda}_\alpha^{p,\infty} \subseteq \dot{\Lambda}_{\alpha-\frac{d}{p}}$ by Theorem 1.3.4. The proof follows from the previous corollary. □

Theorem 2.2.1. *Suppose $\alpha > 2 + \frac{d}{p}$ and $1 \leq p \leq \infty$. Then K is $\dot{\Lambda}_\alpha^{p,\infty}$ -removable if and only if $K^\circ = \emptyset$ (K° is the interior of K).*

Proof. If $K^\circ \neq \emptyset$ then there exists $f \in C_c^\infty(\mathbb{R}^d)$ such that $f \neq 0$ and $\text{supp}(f) \subseteq K$.

To prove the second part we observe that since $\alpha > 2 + \frac{d}{p}$, any tempered distribution from $\dot{\Lambda}_\alpha^{p,\infty}$ has continuous second order derivatives by the previous corollary and since interior of K is empty it follows that K is $\dot{\Lambda}_\alpha^{p,\infty}$ -removable. \square

An analogous result holds for non-homogeneous Besov spaces too. We can state in general:

Theorem 2.2.2. *Suppose $\alpha > 2 + \frac{d}{p}$ and $1 \leq p \leq \infty$. Then K is $\Lambda_{\alpha,loc}^{p,\infty}$ -removable if and only if $K^\circ = \emptyset$.*

Corollary 2.2.3. *Suppose $\alpha > 2 + \frac{d}{p}$, $1 \leq q \leq \infty$ and $1 \leq p \leq \infty$. Then K is $\Lambda_{\alpha,loc}^{p,q}$ -removable if and only if $K^\circ = \emptyset$.*

Proof. The proof easily follows from the fact that $\Lambda_{\alpha,loc}^{p,q} \subset \Lambda_{\alpha,loc}^{p,\infty}$ when $\alpha > 0$. \square

The next result considers the case when $\alpha > 2$ but it gives only sufficient condition.

We start with next

Lemma 2.2.3. *Suppose $\alpha > 0$ and $1 \leq p \leq \infty$. Then:*

- (i) $\Lambda_\alpha^{p,\infty} \subseteq L^p$;
- (ii) $\dot{\Lambda}_\alpha^{p,\infty} \subseteq L_{loc}^p$.

Proof. (i) Suppose $f \in \Lambda_\alpha^{p,\infty}$, and $f = F + \sum_{n=0}^{\infty} f_n$, then $F \in L^p$ and $\|f_n\|_p \leq C2^{-n\alpha}$ which implies $\sum_{n=0}^{\infty} \|f_n\|_p < \infty$ since $0 < \alpha$.

(ii) Suppose $f \in \dot{\Lambda}_\alpha^{p,\infty}$, and $f = \sum_{i=-\infty}^{\infty} f_n$. Then $\sum_{n=0}^{-\infty} f_n \in C^\infty$ and $\sum_{n=0}^{\infty} f_n \in L^p$, so $\dot{\Lambda}_\alpha^{p,\infty} \subseteq L_{loc}^p$. \square

Theorem 2.2.3. *Suppose $\alpha > 2$ and $1 \leq p \leq \infty$. If $m_d(K) = 0$ then K is $\Lambda_{\alpha,loc}^{p,\infty}$ removable.*

Proof. Say $f \in \Lambda_{\alpha,loc}^{p,\infty}$, f harmonic in $\mathbb{R}^d \setminus K$, $m_d(K) = 0$. Then $\Delta f \in L_{loc}^p$, $\Delta f = 0$ a.e., so $\Delta f = 0$. \square

Corollary 2.2.4. *Suppose $\alpha > 2$, $1 \leq q \leq \infty$ and $1 \leq p \leq \infty$. If $m_d(K) = 0$ then K is $\Lambda_{\alpha,loc}^{p,q}$ removable.*

Proof. The proof easily follows from the fact that $\Lambda_{\alpha,loc}^{p,q} \subset \Lambda_{\alpha,loc}^{p,\infty}$ when $\alpha > 0$. \square

The converse of this theorem is true in the case when $\alpha = 2$ but in the general case we do not know the answer.

Lemma 2.2.4. *Suppose $1 \leq p \leq \infty$. If $f \in L^p$ then $f * g \in \mathring{\Lambda}_2^{p,\infty}$ where g is the Green function (see definition 1.2.1).*

Proof. We know that $\Delta(f * g) = f \in \mathring{\Lambda}_0^{p,\infty}$ so by Theorem 1.3.2 we have $f * g \in \mathring{\Lambda}_2^{p,\infty}$. \square

Theorem 2.2.4. *Suppose K is a compact subset of \mathbb{R}^d and $1 \leq p \leq \infty$. If $m_d(K) > 0$ then K is not $\Lambda_{2,loc}^{p,\infty}$ -removable.*

Proof. Since $m_d(K) > 0$ there exists $f \in L^p$, such that $\int_{\mathbb{R}^d} f(x) dx = 1$ and $\text{supp}(f) \subseteq K$. Then by Lemma 2.2.4 $f * g \in \Lambda_{2,loc}^{p,\infty}$. So K is not $\Lambda_{2,loc}^{p,\infty}$ -removable. \square

Now we turn our attention to the lower bound of the parameter α .

Lemma 2.2.5. *$g \in \mathring{\Lambda}_{2-\frac{d}{p'}}^{p,\infty}$ for any $p \in [1, \infty]$.*

Proof. We consider the case when $d > 2$. The other cases are very similar to prove.

By Lemma 1.3.4

$$S_r(g)(x) = \begin{cases} \frac{-1}{A_1(d-2)}(r^{2-d} - |x|^{2-d}), & \text{if } |x| \leq r, \\ 0, & \text{otherwise} \end{cases}.$$

Since $2 - \frac{d}{p'} < 2$ we may use Theorem 1.3.7 to finish the proof:

$$\begin{aligned} \frac{\|S_r g\|_p^p}{r^{(2-\frac{d}{p'})p}} &\leq C \frac{\int_{B(0,r)} |x|^{(2-d)p} dx}{r^{(2-\frac{d}{p'})p}} \\ &\leq C r^{(2-d)p+d-(2-\frac{d}{p'})p} \\ &= C \end{aligned}$$

□

Corollary 2.2.5. *Suppose $\alpha = 2 - \frac{d}{p'}$ and $p \in [1, \infty]$. Then K is $\mathring{\Lambda}_\alpha^{p,\infty}$ -removable if and only if $K = \emptyset$.*

Recall that if $\gamma_1 < \gamma_2$ then $\mathring{\Lambda}_{\gamma_2,loc}^{p,\infty} \subset \mathring{\Lambda}_{\gamma_1,loc}^{p,\infty}$ and $\mathring{\Lambda}_\gamma^{p,\infty} \subset \Lambda_{\gamma,loc}^{p,\infty}$, which implies

Corollary 2.2.6. *Suppose $\alpha \leq 2 - \frac{d}{p'}$ and $1 \leq p \leq \infty$. Then K is $\Lambda_{\gamma,loc}^{p,\infty}$ -removable if and only if $K = \emptyset$.*

So the only interesting case is $\alpha > 2 - \frac{d}{p'}$.

2.3 Constructing Functions in Besov Spaces Using Measures

Proposition 2.3.1. *Suppose that μ is a measure on \mathbb{R}^d and there exists a constant $C > 0$ such that $\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} < C$ for $0 < r \leq 1$ (μ_r is given by definition 1.2.3). Then $\|S_r(u)\|_p < Cr^\alpha$ for $0 < r \leq 1$, where $u = \mu * g$ (g is the Green's function).*

Proof. It is clear that

$$S_r(u) = \mu * S_r(g) = \mu * \left(\frac{1}{A_1} \int_0^r t^{1-d} \mathbf{1}_{B(0,t)} dt \right) = \frac{1}{A_1} \int_0^r t^{1-d} \mu * \mathbf{1}_{B(0,t)} dt = \frac{1}{A_1} \int_0^r t^{1-d} \mu_t dt$$

and so, by Minkowski's inequality,

$$\begin{aligned} \|S_r(u)\|_p &\leq \int_0^r t^{1-d} \|\mu_t\|_p dt \\ &\leq C \int_0^r t^{1-d} t^{d-2+\alpha} dt \\ &= C \int_0^r t^{-1+\alpha} dt \\ &= Cr^\alpha. \end{aligned}$$

□

Proposition 2.3.2. *Suppose $p \in [1, \infty]$, $q \in [1, \infty)$, $0 < \alpha < 2$, and μ is a measure on \mathbb{R}^d . If $\int_0^\infty \left(\frac{\|\mu_r\|_p}{r^{\alpha+d-2}} \right)^q \frac{dr}{r} < \infty$ then $\mu * g \in \dot{\Lambda}_\alpha^{p,q}$.*

Proof. We know that $S_r(\mu * g) = \frac{1}{A_1} \int_0^r t^{1-d} \mu_t dt$. So $\|S_r(\mu * g)\|_p \leq C \int_0^r t^{1-d} \|\mu_t\|_p dt$.

Suppose $\int_0^\infty \left(\frac{\|\mu_r\|_p}{r^{\alpha+d-2}} \right)^q \frac{dr}{r} < \infty$. Then Theorem 1.2.2 implies:

$$\begin{aligned}
\int_0^\infty \left(\frac{\|S_r(\mu * g)\|_p}{r^\alpha} \right)^q \frac{dr}{r} &\leq C \int_0^\infty r^{-\alpha q - 1} \left(\int_0^r t^{1-d} \|\mu_t\|_p dt \right)^q dr \\
&\leq C \int_0^\infty r^{q-\alpha q - 1} (r^{1-d} \|\mu_r\|_p)^q dr \\
&= C \int_0^\infty \left(\frac{\|\mu_r\|_p}{r^{\alpha+d-2}} \right)^q \frac{dr}{r} \\
&< \infty.
\end{aligned}$$

Thus by Theorem 1.3.7 we have $\mu * g \in \mathring{\Lambda}_\alpha^{p,q}$. \square

Proposition 2.3.3. *Suppose μ is a measure on \mathbb{R}^d , $0 < \alpha < 2$. If $p \in [1, \infty]$, $q \in [1, \infty)$, and $\int_0^1 \left(\frac{\|\mu_r\|_p}{r^{\alpha+d-2}} \right)^q \frac{dr}{r} < \infty$ then $\mu * g \in \Lambda_\alpha^{p,q}$.*

The proof is exactly the same as of Proposition 2.3.2. To prove the converse of Propositions 2.3.1 and 2.3.2 we need next

Lemma 2.3.1. *Suppose μ is a measure on \mathbb{R}^d and $\alpha > 0$. If $\|\mu_r\|_p > Mr^{d+\alpha-2}$ for some $M > 0$ and $r > 0$ then there exists $C > 0$ such that $\|S_{2r}(g * \mu)\|_p > CMr^\alpha$.*

Proof. We are considering the case $d > 2$. Cases $d = 1$ and $d = 2$ are very similar.

We know that

$$S_r(g)(x) = \begin{cases} \frac{-1}{(d-1)A_1} (r^{2-d} - |x|^{2-d}), & \text{if } x \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

where A_1 is the area of the unit sphere. This implies that

$$|S_{2r}(g)(x)| \geq \frac{1}{(d-1)A_1} \left(\frac{1}{r^{d-2}} - \frac{1}{(2r)^{d-2}} \right) = C \frac{1}{r^{d-2}}$$

when $|x| \leq r$. Then

$$\begin{aligned} \|S_{2r}(g * \mu)\|_p &= \|S_{2r}(g) * \mu\|_p \\ &\geq \frac{C}{r^{d-2}} \|\mathbf{1}_{B(0,r)} * \mu\|_p \\ &= Cr^{-d+2} \|\mu_r\|_p \\ &\geq Cr^{-d+2+d-2+\alpha} M \\ &= CMr^\alpha. \end{aligned}$$

□

Corollary 2.3.1. *Suppose μ is a measure in \mathbb{R}^d supported on the compact set K and $\alpha > 0$. If for any $M > 0$ there exists $r > 0$ such that $\|\mu_r\|_p \geq Mr^{d-2+\alpha}$ then for any $M_1 > 0$ there exists $r_1 > 0$ such that $\|S_{r_1}(\mu * g)\|_p > M_1 r^\alpha$*

Proof. This is clear from the previous lemma. □

Corollary 2.3.2. *Suppose μ is measure in \mathbb{R}^d supported on the compact set K . If*

$$\int_0^1 \left(\frac{\|\mu_r\|_p}{r^{d-2+\alpha}} \right)^q \frac{dr}{r} = \infty \text{ then } \int_0^1 \left(\frac{\|S_r(u)\|_p}{r^\alpha} \right)^q \frac{dr}{r} = \infty \text{ where } u = \mu * g.$$

Proof.

$$\begin{aligned}
\int_0^1 \left(\frac{\|S_r(u)\|_p}{r^\alpha} \right)^q \frac{dr}{r} &\geq C \int_0^1 \left(\frac{\|\mu_{\frac{r}{2}}\|_p r^{-d+2}}{r^\alpha} \right)^q \frac{dr}{r} \\
&= C \int_0^1 \left(\frac{\|\mu_{\frac{r}{2}}\|_p}{r^{\alpha-2+d}} \right)^q \frac{dr}{r} \\
&= C \int_0^{\frac{1}{2}} \left(\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} \right)^q \frac{dr}{r} \\
&= \infty,
\end{aligned}$$

since $\int_{\frac{1}{2}}^1 \left(\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} \right)^q \frac{dr}{r} < \infty$. □

2.4 β and Non-Removable Sets

Theorem 2.4.1. *Suppose μ is a probability measure supported on a compact set $K \subset \mathbb{R}^d$.*

(i) *If $\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} \leq C$ for some constant $C > 0$ when $0 < r \leq 1$, $1 \leq p \leq \infty$ and $0 < \alpha < 2$, then $\mu * g \in \Lambda_\alpha^{p,\infty}$ and $\mu * g$ has a non-removable singularity on K .*

(ii) *If $\int_0^1 \left(\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} \right)^q \frac{dr}{r} < \infty$ ($0 < \alpha$, $1 \leq p \leq \infty$, $1 \leq q < \infty$) then $\mu * g \in \Lambda_\alpha^{p,q}$ and $\mu * g$ has a non-removable singularity on K .*

Proof. (i) We have already shown in proving Proposition 2.3.1 that under given conditions we have $\|\mu * S_r(g)\|_p \leq Cr^\alpha$. By Theorem 1.3.7 it follows that $\mu * g \in \Lambda_\alpha^{p,\infty}$.

On the other hand $|\mu * g(x)| = \left| \int_{\mathbb{R}^d} g(x-y) d\mu(y) \right| \leq C \frac{\mu(K)}{(|x| - \text{diam}(K))^{d-2}}$ (when $|x|$ is big enough) which shows that $\mu * g(x) \rightarrow 0$ when $|x| \rightarrow \infty$. But $\mu * g$ is harmonic outside of K and $\Delta(\mu * g) = \mu * \Delta g = \mu * \delta_0 = \mu \neq 0$ (as a distribution). Therefore $\mu * g$ cannot be extended to a harmonic function in \mathbb{R}^d .

(ii) We know from proposition 2.3.3 that $\mu * g \in \Lambda_\alpha^{p,q}$ and an argument similar to the final argument of part (i) finishes the proof. \square

Similarly we can prove the following

Theorem 2.4.2. *Suppose μ is a probability measure supported on a compact set $K \subset \mathbb{R}^d$. If $\int_0^\infty \left(\frac{\|\mu_r\|_p}{r^{\alpha-2+d}} \right)^q \frac{dr}{r} < \infty$ ($1 \leq q < \infty$, $0 < \alpha < 2$) then $\mu * g \in \dot{\Lambda}_\alpha^{p,q}$ and $\mu * g$ has a non-removable singularity on the K .*

Now we can characterize non removable singularities for $\Lambda_\alpha^{p,\infty}$ -spaces:

Theorem 2.4.3. *Suppose K is a compact subset of \mathbb{R}^d , $0 < \alpha < 2$, and $1 \leq p \leq \infty$. If $m_\beta(K) > 0$, where $\beta = (\alpha - 2)p' + d$, then K is not $\Lambda_\alpha^{p,\infty}$ removable.*

Proof. Lemma 1.2.1 implies that there exists a probability measure μ supported on the K such that $\|\mu_r\|_\infty \leq Cr^\beta$. Since $\|\mu_r\|_1 = Cr^d\mu(K)$, we have

$$\|\mu_r\|_p^p \leq \|\mu_r\|_\infty^{p-1} \|\mu_r\|_1 \leq Cr^{(p-1)\beta+d} = Cr^{p(\alpha+d-2)}.$$

or

$$\|\mu_r\|_p \leq cr^{\alpha-2+d}.$$

Note that $(p-1)\beta+d = (p-1)((\alpha-2)p'+d)+d = (p-1)p'(\alpha-2)+dp = (\alpha-2+d)p$ since $(p-1)p' = p$. Now we apply Theorem 2.4.1 which finishes the proof. \square

It follows that if $m_\beta(K) > 0$ then K is not $\Lambda_{\alpha,loc}^{p,\infty}$ removable.

Corollary 2.4.1. *Suppose $0 < \alpha < 2$, $1 \leq q \leq \infty$, and $1 \leq p \leq \infty$. If there exists $\gamma > \beta$ such that $m_\gamma(K) > 0$ then K is not $\Lambda_{\alpha,loc}^{p,q}$ -removable.*

Proof. There exists $0 < \alpha_1 < 2$ such that $\gamma = (\alpha_1 - 2)p' + d$. Since $\beta < \gamma$ we have $\alpha < \alpha_1$ which implies that $\Lambda_{\alpha_1, loc}^{p, \infty} \subset \Lambda_{\alpha, loc}^{p, q}$. We apply the previous theorem to finish the proof. \square

3 Conditions of Removability

3.1 Duality in Besov Spaces

Proposition 3.1.1. *Suppose $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then for any $0 < \alpha$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$ there exists $C = C(\alpha, \varphi)$ such that*

$$\|\varphi f\|_{p,q,\alpha} \leq C \|f\|_{p,q,\alpha}$$

for all $f \in \Lambda_\alpha^{p,q}$.

Proof. Suppose f is a distribution from $\Lambda_\alpha^{p,q}$. Then the first part of theorem 1.3.6 implies that f can be presented as $f = \sum_{k \in \mathbb{Z}^d} s_k b_k + \sum_{\nu=0}^{\infty} \sum_{l(Q)=2^{-\nu}} s_Q a_Q$, where the a_Q 's are (α, p) -atoms, and the b_k 's satisfy $\text{supp}(b_k) \subseteq 3Q_{0k}$ and $|D^\eta b_k(x)| \leq 1$ if $|\eta| \leq K$. The convergence is in \mathcal{S}' , and

$$\left(\sum_{k \in \mathbb{Z}^d} |s_k|^p \right)^{\frac{1}{p}} + \left(\sum_{\nu=0}^{\infty} \left(\sum_{l(Q)=2^{-\nu}} |s_Q|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \|f\|_{p,q,\alpha},$$

with C independent of f . Since $\varphi \in \mathcal{S}$ and K is a fixed positive integer it is clear that there exists A such that $\|D^\eta \left(\frac{\varphi b_k}{A} \right)\|_\infty \leq 1$ when $|\eta| \leq K$. Since $0 < \alpha$ and $1 \leq p$ we do not worry about vanishing moments for $\frac{\varphi a_Q}{A}$ and so $\frac{\varphi a_Q}{A}$ is an atom. We write φf as $\varphi f = \sum_{k \in \mathbb{Z}^d} A s_k \frac{\varphi b_k}{A} + \sum_{\nu=0}^{\infty} \sum_{l(Q)=2^{-\nu}} A s_Q \frac{\varphi a_Q}{A}$. So we can use the second part of theorem 1.3.6 to conclude that $\|\varphi f\|_{p,q,\alpha} \leq C \|f\|_{p,q,\alpha}$. \square

Lemma 3.1.1. *Suppose f is a tempered distribution. Then $f \in \Lambda_{\alpha,loc}^{p,q}$ if and only if for any open bounded set $\Omega \in \mathbb{R}^d$ there exists $\tilde{f} \in \Lambda_{\alpha}^{p,q}$ such that $f = \tilde{f}$ in Ω .*

Proof. Suppose $f \in \Lambda_{\alpha,loc}^{p,q}$ and Ω is an open, bounded set in \mathbb{R}^d . Consider a function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(x) = 1$ when $x \in \Omega$. Then by Proposition 3.1.1 we can take $\tilde{f} = f\psi$.

To prove the second part, assume that f is a tempered distribution and ψ is an arbitrary function from $C_c^\infty(\mathbb{R}^d)$. Consider a bounded open set $\Omega \in \mathbb{R}^d$ such that $\text{supp}(\psi) \subset \Omega$. Then there exists $\tilde{f} \in \Lambda_{\alpha}^{p,q}$ with $f = \tilde{f}$ in Ω . So by Proposition 3.1.1 $f\psi = \tilde{f}\psi \in \Lambda_{\alpha}^{p,q}$. \square

Lemma 3.1.2. *There exists $C > 0$ such that for any function $h \in L^p$ which satisfies the condition $\text{supp}(\hat{h}) \subset \mathbb{A}_n$, we have $\|g * h\|_p \leq C2^{-2n}\|h\|_p$.*

Proof. Let us consider a function $\varphi_0 \in \mathcal{S}$ such that $\hat{\varphi}_0 \in C_c^\infty$ and $\hat{\varphi}_0(\xi) = \frac{-1}{4\pi^2|\xi|^2}$ when $\xi \in \mathbb{A}_0$. We define $\varphi_n(x) = 2^{-2n+nd}\varphi_0(2^n x)$. Then

$$\hat{\varphi}_n(\xi) = 2^{-2n+nd-nd}\hat{\varphi}_0(2^{-n}\xi) = \frac{-2^{-2n}}{4\pi^2 2^{-2n}|\xi|^2} = \frac{-1}{4\pi^2|\xi|^2}, \quad (\xi \in \mathbb{A}_n).$$

So $\hat{\varphi}_n$ coincides with \hat{g} on the annulus \mathbb{A}_n (see the remark on page 4). Thus

$$\|g * h\|_p = \|\varphi_n * h\|_p \leq \|\varphi_n\|_1 \|h\|_p \leq C2^{-2n}\|\varphi_0\|_1 \|h\|_p \leq C2^{-2n}\|h\|_p.$$

\square

Lemma 3.1.3. *Suppose $f \in \Lambda_{\alpha}^{p,q}$ where $-\infty < \alpha < \infty$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.*

*Then $(S_r g) * f \in \Lambda_{\alpha+2}^{p,q}$.*

Proof. $\Psi * ((S_r g) * f) = (S_r g) * (\Psi * f) \in L^p$, since $S_r g \in L^1$ and $\Psi * f \in L^p$.

And

$$\begin{aligned} \|\Phi_n * ((S_r g) * f)\|_p &= \|S_r(g * (\Phi_n * f))\|_p \\ &\leq C\|g * (\Phi_n * f)\|_p \end{aligned}$$

But $\text{supp } (\Phi_n * f)^\wedge \subset \mathbb{A}^n$ which by Lemma 3.1.2 implies that

$$\|g * (\Phi_n * f)\|_p \leq C2^{-2n}\|\Phi_n * f\|_p.$$

So

$$\sum_{n=0}^{\infty} (2^{n(\alpha+2)}\|\Phi_n * ((S_r g) * f)\|)^q \leq C \sum_{n=0}^{\infty} (2^{n\alpha}\|\Phi_n * f\|_p)^q < \infty.$$

□

Proposition 3.1.2. *If $f \in \Lambda_\alpha^{p,q}$ ($1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$) and f has compact support then $g * f \in \Lambda_{\alpha+2,loc}^{p,q}$.*

Proof. By Lemma 3.1.1, we need to show that if Ω is a bounded open set then there exists $h \in \Lambda_{\alpha+2}^{p,q}$ with $h = g * f$ in Ω .

It is clear that for any $r > 0$ there exists $C_r > 0$ such that $g(x) = S_r g(x) + C_r$ for $|x| < r$. Since Ω is bounded and f has compact support, if r is large enough then $f * g = f * (S_r g + C_r)$ in Ω . But $f * C_r$ is constant, so there exists $\psi \in C_c^\infty$ with $f * g = f * S_r g + \psi$ in Ω . Since by Lemma 3.1.3 $f * S_r g \in \Lambda_{\alpha+2}^{p,q}$ and $\psi \in \Lambda_{\alpha+2}^{p,q}$ we have $f * g \in \Lambda_{\alpha+2,loc}^{p,q}$. □

Theorem 3.1.1. *Suppose K is a compact subset of \mathbb{R}^d , $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. K is not $\Lambda_{\alpha,loc}^{p,q}$ -removable if and only if there exists $h \in \Lambda_{\alpha-2}^{p,q}$ such that $h \neq 0$, $\text{supp}(h) \subseteq K$.*

Proof. Suppose K is not $\Lambda_{\alpha,loc}^{p,q}$ -removable. So there exists $f \in \Lambda_{\alpha,loc}^{p,q}$ such that f is harmonic in $\mathbb{R}^d \setminus K$ but not harmonic in \mathbb{R}^d . Let $h = \Delta f$. Then it is clear that $\text{supp}(h) \subseteq K$, $h \neq 0$ and $h \in \Lambda_{\alpha-2}^{p,q}$.

To prove the other half of this theorem, assume that there exists $h \in \Lambda_{\alpha-2}^{p,q}$, such that $\text{supp}(h) \subseteq K$, $h \neq 0$. Let $f = g * h$. Then $\Delta f = h$, so f is harmonic in $\mathbb{R}^d \setminus K$, not harmonic in \mathbb{R}^d , and by Proposition 3.1.1 we have $f \in \Lambda_{\alpha,loc}^{p,q}$. \square

The next theorem is an unpublished result of D. Ullrich and by courtesy of the author we are providing the proof.

Theorem 3.1.2. *Suppose K is a compact subset of \mathbb{R}^d , $\alpha < 0$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Then the following are equivalent:*

- (a) *If $h \in \Lambda_{\alpha}^{p,q}$ and $\text{supp}(h) \subseteq K$ then $h = 0$;*
- (b) *For any $\epsilon > 0$ there exists $\psi_{\epsilon} \in \Lambda_{-\alpha}^{p',q'}$ such that $\psi_{\epsilon} = 1$ on a neighborhood of K and $\|\psi_{\epsilon}\|_{p',q',-\alpha} < \epsilon$.*

Proof. First we show that (b) implies (a): Suppose $h \in \Lambda_{\alpha}^{p,q}$ and $\text{supp}(h) \subseteq K$. Consider any $\epsilon > 0$ and $\psi_{\epsilon} \in \Lambda_{-\alpha}^{p',q'}$ such that $\psi_{\epsilon} = 1$ on a neighborhood of K and $\|\psi_{\epsilon}\|_{p',q',-\alpha} < \epsilon$. For any $\varphi \in C_c^{\infty}$, we have $\psi_{\epsilon}\varphi = \varphi$ on a neighborhood of $\text{supp}(h)$, so

from Proposition 3.1.1 follows that

$$\begin{aligned}
|\langle h, \varphi \rangle| &= |\langle h, \psi_\epsilon \varphi \rangle| \\
&\leq \|h\|_{p,q,\alpha} \|\psi_\epsilon \varphi\|_{p',q',-\alpha} \\
&\leq C \|h\|_{p,q,\alpha} \|\psi_\epsilon\|_{p',q',-\alpha} \\
&\leq C \|h\|_{p,q,\alpha} \epsilon.
\end{aligned}$$

Since ϵ is an arbitrary positive number, this implies that $\langle h, \phi \rangle = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, which means $h = 0$.

To prove the second part of this theorem we assume that (a) is right and (b) is false, which will lead us to contradiction. Suppose there exists $M > 0$ such that for any $\psi \in \Lambda_{-\alpha}^{p',q'}$ and $\psi = 1$ on a neighborhood of K we have $\|\psi\|_{p',q',-\alpha} \geq M$. Note that this implies $K \neq \emptyset$.

Let $\mathbb{X} = \left\{ \psi \in \Lambda_{-\alpha}^{p',q'} : \psi \in \mathcal{S} \text{ and } \psi \text{ is constant in a neighborhood of } K \right\}$. Fix $x \in K$. Define $\lambda : \mathbb{X} \rightarrow \mathbb{C}$ by $\lambda(\psi) = \psi(x)$.

It is clear that $|\lambda(\psi)| \leq \frac{1}{M} \|\psi\|_{p',q',-\alpha}$ for all $\psi \in \mathbb{X}$. The Hahn-Banach theorem [4] implies that λ extends to $\Lambda_{-\alpha,0}^{p',q'}$ with the same norm. By Theorem 1.3.5 there exists $f \in \Lambda_{\alpha}^{p,q}$ such that $\langle f, \psi \rangle = \lambda(\psi)$ for any $\psi \in \Lambda_{-\alpha,0}^{p',q'}$. Hence $\langle f, \psi \rangle = \psi(x)$ ($x \in K, \psi \in \mathbb{X}$).

Now $\lambda \neq 0$ implies that $f \neq 0$. We need to show that $\text{supp}(f) \subseteq K$. This is equivalent of showing that $\langle f, \varphi \rangle = 0$ for all $\varphi \in C_c^\infty$, $\text{supp}(\varphi) \subseteq \mathbb{R}^d \setminus K$.

But $\text{supp}(\varphi) \subseteq \mathbb{R}^d \setminus K$ implies that $\varphi = 0$ in a neighborhood of K , so $\varphi \in \mathbb{X}$ and $\langle f, \varphi \rangle = \varphi(x) = 0$. □

Theorem 3.1.3. *Suppose K is a compact subset of \mathbb{R}^d and $\alpha < 2$. K is $\Lambda_{\alpha,loc}^{p,q}$ -removable if and only if for any $\epsilon > 0$ there exists $\psi_\epsilon \in \Lambda_{2-\alpha}^{p',q'}$, $\psi_\epsilon = 1$ on a neighborhood of K , and $\|\psi_\epsilon\|_{p',q',2-\alpha} < \epsilon$.*

Proof. This follows from theorems 3.1.1 and 3.1.2. □

3.2 β and Removable Sets

We will prove the following lemma and theorem in case when $d = 2$. They can be generalized in higher dimensional Euclidean spaces easily and the proof is very similar.

We introduce following notations: $Q_n = [-2^{n-1}, 2^{n-1}] \times [-2^{n-1}, 2^{n-1}]$, for $n \in \mathbb{Z}$.

By \mathbb{A}_n^\diamond , $n \in \mathbb{Z}$, we denote $Q_n \setminus Q_{n-1}$.

Proposition 3.2.1. *There exists $C > 0$ such that for any $h = \sum_{n=-\infty}^{\infty} h_n$ where $\text{supp}(\hat{h}) \subset \mathbb{A}_n^\diamond$ and $\left(\sum_{n=-\infty}^{\infty} (2^{\alpha n} \|h_n\|_p)^q \right)^{\frac{1}{q}} < \infty$ ($1 \leq p \leq \infty$, $1 \leq q \leq \infty$) we have*

$$\|h\|_{p,q,\alpha}^\diamond \leq C \left(\sum_{n=-\infty}^{\infty} (2^{\alpha n} \|h_n\|_p)^q \right)^{\frac{1}{q}} < \infty.$$

Proof. Since $\mathbb{A}_n = \{\xi : 2^{n-1} \leq |\xi| \leq 2^{n+1}\}$ it is clear that

$$\mathbb{A}_n \subset \mathbb{A}_{n-1}^\diamond \cup \mathbb{A}_n^\diamond \cup \mathbb{A}_{n+1}^\diamond \cup \mathbb{A}_{n+2}^\diamond.$$

So $h * \Phi_n = (h_{n-1} + h_n + h_{n+1} + h_{n+2}) * \Phi_n$. Because of this

$$\begin{aligned} \|h * \Phi_n\|_p &\leq \|\Phi_n\|_1 (\|h_{n-1}\|_p + \|h_n\|_p + \|h_{n+1}\|_p + \|h_{n+2}\|_p) \\ &= \|\Phi_0\|_1 (\|h_{n-1}\|_p + \|h_n\|_p + \|h_{n+1}\|_p + \|h_{n+2}\|_p). \end{aligned}$$

All of this implies:

$$\begin{aligned}
\|h\|_{p,q,\alpha}^\circ &= \left(\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|\Phi_n * (h_{n-1} + h_n + h_{n+1} + h_{n+2})\|_p)^q \right)^{\frac{1}{q}} \\
&\leq \left(\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|\Phi_0\|_1 (\|h_{n-1}\|_p + \|h_n\|_p + \|h_{n+1}\|_p + \|h_{n+2}\|_p))^q \right)^{\frac{1}{q}} \\
&\leq \|\Phi_0\| \left(\sum_{n=-\infty}^{\infty} (2^{qn\alpha} 4^q (\|h_{n-1}\|_p^q + \|h_n\|_p^q + \|h_{n+1}\|_p^q + \|h_{n+2}\|_p^q)) \right)^{\frac{1}{q}} \\
&= 4\|\Phi_0\| \left(\sum_{n=-\infty}^{\infty} (2^{qn\alpha} \|h_{n-1}\|_p^q + 2^{qn\alpha} \|h_n\|_p^q + 2^{qn\alpha} \|h_{n+1}\|_p^q + 2^{qn\alpha} \|h_{n+2}\|_p^q) \right)^{\frac{1}{q}} \\
&\leq 4\|\Phi_0\|_1 \sum_{i=-1}^3 \left(\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|h_{n+i}\|_p)^q \right)^{\frac{1}{q}} \\
&= 4\|\Phi_0\|_1 (2^\alpha + 1 + 2^{-\alpha} + 2^{-2\alpha}) \left(\sum_{n=-\infty}^{\infty} (2^{\alpha n} \|h_n\|_p)^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus we take $C = 4\|\Phi_0\|_1(2^\alpha + 1 + 2^{-\alpha} + 2^{-2\alpha})$ and we are done. \square

Lemma 3.2.1. Assume $Q = [-r, r] \times [-r, r]$, $f(x) = \mathbf{1}_{[-r, r]}(x)$, $g(x_1, x_2) = f(x_1)f(x_2)$.

Then there exists $C > 0$ such that $\|f * \Phi_n\|_p \leq C2^{-n}2^{\frac{n}{p'}}\|\Phi_0\|_p$ if $n > -\log_2(r)$ and $\|f * \Phi_n\|_p \leq Cr2^{\frac{n}{p'}}\|\Phi_0\|_p$ if $n \leq -\log_2(r)$.

Proof. We know that $\hat{f}(\xi) = \frac{\sin(2\pi r\xi)}{\pi\xi}$. We also know that $\|\Phi_n\|_1 = \|\Phi_0\|_1$, $\|\Phi_n\|_\infty = 2^n\|\Phi_0\|_\infty$, $\|\Phi_n\|_p = 2^{\frac{n}{p'}}\|\Phi_0\|_p$ for all $n \in \mathbb{Z}$. Let us introduce a function φ such that $\hat{\varphi}_0(\xi) \in C_c^\infty$ and $\hat{\varphi}_0(\xi) = \frac{1}{\pi\xi}$ when $\xi \in \mathbb{A}_0$. We define $\hat{\varphi}_n(\xi) = 2^{-n}\hat{\varphi}_0(2^{-n}\xi)$ when $n \in \mathbb{N}$. Since $(\tau_r\Phi_n)^\wedge(\xi) = e^{-2\pi r\xi}\hat{\Phi}_n(\xi)$ we have an equality

$$f * \Phi_n = (-\tau_r\Phi_n - \tau_{-r}\Phi_n) * \varphi_n$$

which implies that

$$\begin{aligned}
\|f * \Phi_n\|_p &= \|(-\tau_r \Phi_n - \tau_{-r} \Phi_n) * \varphi_n\|_p \\
&\leq C \|\varphi_n\|_1 \|\Phi_n\|_p \\
&= C 2^{-n} \|\varphi_0\|_1 \|\Phi_n\|_p \\
&= C 2^{-n} 2^{\frac{n}{p'}} \|\Phi_0\|_p.
\end{aligned}$$

This finishes the first part of the lemma. The second part is easy:

$$\|f * \Phi_n\|_p \leq \|f\|_1 \|\Phi_n\|_p \leq C r 2^{\frac{n}{p'}} \|\Phi_0\|_1.$$

□

Theorem 3.2.1. *Suppose $-\frac{d}{p'} < \alpha < \frac{1}{p}$, $1 < p < \infty$, $1 \leq q < \infty$. Then homogeneous Besov space $\dot{\Lambda}_\alpha^{p,q}$ contains the characteristic function of a d -dimensional cube Q and there exists $C > 0$ such that*

$$\|\mathbf{1}_Q\|_{p,q,\alpha}^\circ \leq C l(Q)^{-\alpha + \frac{d}{p}}$$

where $l(Q)$ is the length of cube Q . If $q = \infty$ then we may assume $-\frac{d}{p'} \leq \alpha \leq \frac{1}{p}$.

Proof. Recall that we are considering the case when $d = 2$ and we are using following notations: $Q = [-r, r] \times [-r, r]$, $f(x) = \mathbf{1}_{[-r,r]}(x)$, $g(x_1, x_2) = f(x_1)f(x_2)$. If $f =$

$\sum_{n=-\infty}^{\infty} f_n$ ($f_n = f * \Phi_n$) then $g = \sum_{n=-\infty}^{\infty} g_n$ where

$$g_n(x_1, x_2) = f_n(x_1)f_n(x_2) + \sum_{i=-\infty}^{n-1} (f_n(x_1)f_i(x_2) + f_i(x_1)f_n(x_2)).$$

It is clear that

$$\text{supp}(\hat{g}_n) \subseteq \left(\text{supp}(f_n) \times \bigcup_{i=-\infty}^n \text{supp}(f_i) \right) \cup \left(\left(\bigcup_{i=-\infty}^n \text{supp}(f_i) \right) \times \text{supp}(f_n) \right) \subset \mathbb{A}_n^{\diamond}.$$

By proposition 3.2.1 it is enough to show that $\sum_{n=-\infty}^{\infty} (2^{n\alpha} \|g_n\|_p)^q \leq Cl(Q)^{-\alpha q + \frac{qd}{p}}.$

By N we denote the greatest integer which is smaller or equal to $-\log_2(r)$ ($2^N \leq \frac{1}{r} < 2^{N+1}$).

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (2^{n\alpha} \|g_n\|_p)^q &= \sum_{n=-\infty}^{\infty} \left(2^{n\alpha} \|f_n(x_1)f_n(x_2) + \sum_{i=-\infty}^{n-1} (f_n(x_1)f_i(x_2) + f_i(x_1)f_n(x_2))\|_p \right)^q \\ &\leq \sum_{n=-\infty}^{\infty} \left(2^{n\alpha} \sum_{i=-\infty}^n (2\|f_n\|_p\|f_i\|_p) \right)^q \\ &= 2^q \sum_{n=-\infty}^N \left(2^{n\alpha} \sum_{i=-\infty}^n (\|f_n\|_p\|f_i\|_p) \right)^q + 2^q \sum_{n=N+1}^{\infty} \left(2^{n\alpha} \sum_{i=-\infty}^n (\|f_n\|_p\|f_i\|_p) \right)^q \\ &= 2^q \sum_{n=-\infty}^N \left(2^{n\alpha} \|f_n\|_p \sum_{i=-\infty}^n \|f_i\|_p \right)^q + 2^q \sum_{n=N+1}^{\infty} \left(2^{n\alpha} \|f_n\|_p \sum_{i=-\infty}^n \|f_i\|_p \right)^q. \end{aligned}$$

Let us estimate the last two terms separately:

$$\begin{aligned}
\sum_{n=-\infty}^N \left(2^{n\alpha} \|f_n\|_p \sum_{i=-\infty}^n \|f_i\|_p \right)^q &\leq \sum_{n=-\infty}^N \left(2^{n\alpha} \|f_n\|_p \sum_{i=-\infty}^n 2^{\frac{i}{p'}} r \|\Phi_0\|_p \right)^q \\
&\leq \sum_{n=-\infty}^N \left(2^{n\alpha} \|f_n\|_p C 2^{\frac{n}{p'}} r \right)^q \\
&\leq C r^q \sum_{n=-\infty}^N \left(2^{n\alpha} r 2^{\frac{n}{p'}} 2^{\frac{n}{p'}} \right)^q \\
&\leq C r^{2q} 2^{N(\alpha+2\frac{1}{p'})q} \\
&= C r^{(2-\alpha-\frac{2}{p'})q} \\
&= C r^{-q\alpha+\frac{q^2}{p}}.
\end{aligned}$$

Note that the series are convergent when $\alpha + \frac{2}{p'} > 0$ or $\alpha > \frac{-2}{p'}$ ($d = 2$).

Now we estimate the second term:

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \left(2^{n\alpha} \|f_n\|_p \sum_{i=-\infty}^n \|f_i\|_p \right)^q &= \sum_{n=N+1}^{\infty} \left(2^{n\alpha} \|f_n\|_p \left(\sum_{i=-\infty}^N \|f_i\|_p + \sum_{i=N+1}^n \|f_i\|_p \right) \right)^q \\
&\leq \sum_{n=N+1}^{\infty} \left(2^{n\alpha} \|f_n\|_p \left(\sum_{i=-\infty}^N C r 2^{\frac{i}{p'}} + \sum_{i=N+1}^n C 2^{-i} 2^{\frac{i}{p'}} \right) \right)^q \\
&\leq C \sum_{n=N+1}^{\infty} \left(2^{n\alpha} \|f_n\|_p \left(r r^{\frac{-1}{p'}} + 2^{-n} 2^{\frac{n}{p'}} \right) \right)^q \\
&\leq C \sum_{n=N+1}^{\infty} \left(2^{n\alpha} 2^{-n} 2^{\frac{n}{p'}} \left(r r^{\frac{-1}{p'}} + 2^{-n} 2^{\frac{n}{p'}} \right) \right)^q \\
&\leq C \left(2^{N\alpha} 2^{-N} 2^{\frac{N}{p'}} \left(r r^{\frac{-1}{p'}} + 2^{-N} 2^{\frac{N}{p'}} \right) \right)^q \\
&\leq C r^{(-\alpha+1-\frac{1}{p'}+1-\frac{1}{p'})q} \\
&= C r^{(2-\alpha-\frac{2}{p'})q} \\
&= C r^{-q\alpha+\frac{q^2}{p}}.
\end{aligned}$$

We can see that last series converges when $\alpha - 1 + \frac{1}{p'} < 0$ or $\alpha < \frac{1}{p}$. \square

We have an analogous result for the non-homogeneous Besov spaces:

Theorem 3.2.2. *If $-\frac{d}{p'} < \alpha < \frac{1}{p}$ then for any $1 \leq p \leq \infty$, $1 \leq q < \infty$, a non homogeneous Besov space $\Lambda_{\alpha}^{p,q}$ contains a characteristic function of a d -dimensional cube and there exist $C > 0$ (depends on α , p , q) such that*

$$\|\mathbf{1}_Q\|_{p,q,\alpha} \leq C \left(l(Q)^{\frac{d}{p}} + l(Q)^{-\alpha + \frac{d}{p}} \right)$$

If $q = \infty$ then we may assume $-\frac{d}{p'} \leq \alpha \leq \frac{1}{p}$.

Proof. This proof is just a simple modification of the previous proof. \square

Based on this theorem we are giving a criterion of removability property of a compact set for harmonic functions in Besov spaces when the parameter α is between $2 - \frac{1}{p'}$ and 2:

Theorem 3.2.3. *Suppose K is a compact subset of \mathbb{R}^d , $1 \leq p \leq \infty$ and $2 - \frac{1}{p'} < \alpha < 2$. If $m_{\beta}(K) = 0$, where $\beta = (\alpha - 2)p' + d$, then K is $\Lambda_{\alpha}^{p,p}$ -removable.*

Proof. We note that since $2 - \frac{1}{p'} < \alpha < 2$ then $0 < 2 - \alpha < \frac{1}{p'}$ and the previous theorem implies that $\Lambda_{2-\alpha}^{p',p'}$ contains characteristic functions of d -dimensional cubes.

Let $\epsilon > 0$. Since $m_{\beta}(K) = 0$ there exists covering of K by disjoint cubes Q_1, Q_2, \dots, Q_n

such that $\sum_{i=1}^n l(Q_i)^\beta < \epsilon$. Then by theorem 3.2.2 we have:

$$\begin{aligned} \|\mathbf{1}_{\cup_{i=1}^n Q_i}\|_{p',q',2-\alpha} &\leq \|\mathbf{1}_{\cup_{i=1}^n Q_i}\|_{p'} + \left\| \sum_{i=1}^n \mathbf{1}_{Q_i} \right\|_{p',q',2-\alpha}^\circ \\ &\leq \left(\sum_{i=1}^n l(Q_i)^d \right)^{\frac{1}{p'}} + \sum_{i=1}^n \|\mathbf{1}_{Q_i}\|_{p',q',2-\alpha}^\circ \\ &\leq C \left[\left(\sum_{i=1}^n l(Q_i)^d \right)^{\frac{1}{p'}} + \sum_{i=1}^n l(Q_i)^{(\alpha-2)p'+d} \right] \end{aligned}$$

The first term of this sum we can make less than $\frac{\epsilon}{2}$ because when $\alpha \leq 2$ then $\beta = (\alpha - 2)p' + d \leq d$ and so $m_d(K) \leq m_\beta(K)$. Similarly the second term can be made less than $\frac{\epsilon}{2}$ since $m_\beta(K) = 0$. By applying theorem 3.1.3 we finish the proof. \square

The previous result characterizes removable singularities in a special range of parameter α . Next we are giving characterization of singularities in a larger range of α but the required condition is stronger.

Proposition 3.2.2. *There exist $A > 0$ and $C > 0$ such that for any finite set of disjoint dyadic cubes $\Omega = \{Q_1, Q_2, \dots, Q_n\}$ where $l(Q_i) = 2^{-t_i}$, ($t_i \in \mathbb{N}$, $i = 1, \dots, n$), we can construct a function $f = \sum_{i=1}^n b_i$ with properties:*

- (i) $f|_{\cup_{i=1}^n Q_i} = 1$,
- (ii) $\text{supp}(b_i) \subset AQ_i$, $i = 1, \dots, n$,
- (iii) $\|D^\eta(b_i)\|_\infty \leq C 2^{t_i|\eta|} t_i^{|\eta|}$, $i = 1, \dots, n$ for $|\eta| \leq 2$.

Proof. We denote by F_m ($m \in \mathbb{N}$) the set of cubes from Ω with side length equal to 2^{-m} . We choose $A_1 \in \mathbb{N}$ and let \tilde{F}_m be the family of dyadic cubes of side length equal to 2^{-m} which are within A_1 cubes of a cube in F_m . This means that Q is in

\tilde{F}_m if there exist maximum A_1 cubes of side length 2^{-m} such that their union is a connected subset of \mathbb{R}^d and this group contains Q and at least one cube from F_m . We choose a map $f : \tilde{F}_m \rightarrow F_m$ such that: if $Q \in \tilde{F}_m$ then $f(Q)$ is a cube in F_m which is within A_1 cubes from Q and in case when $Q \in F_m$ we make $f(Q) = Q$. For $Q_i \in \Omega$ we introduce the notation $\tilde{Q}_i = \cup_{Q \in f^{-1}(Q_i)} Q$. It is clear that $\tilde{Q}_i \cap \tilde{Q}_k = \emptyset$ if Q_i and Q_k are different cubes from F_m , and we have $\cup_{Q \in F_m} \tilde{Q} = \cup_{Q \in \tilde{F}_m} Q$.

Let us consider $p \in \mathbb{C}_c^\infty$ such that $\text{supp}(p) \subseteq \overline{B(0, 1)}$, $p(x) > 0$ when $x \in B(0, 1)$ and $\int_{\mathbb{R}^d} p(x) dx = 1$. We define $p_m(x) = \left(\frac{2^m}{A_1}\right)^d p\left(\frac{2^m x}{A_1}\right)$ and $\psi_m = \sum_{Q \in \tilde{F}_m} \mathbf{1}_Q * p_m$, for any $m \in \mathbb{N}$. If F_m is empty then we assume that $\psi_m = 0$.

It is clear that $\psi_m = 1$ on Q_i if $Q_i \in F_m$. Now we show that if we choose A_1 big enough then

$$\|D^\eta(\psi_m)\|_\infty \leq 2^{m|\eta|} \quad (|\eta| \leq 2). \quad (3.3)$$

Indeed $\psi_m = h * p_m$ where $\|h\|_\infty = 1$ and $D^\eta(\psi_m) = h * D^\eta(p_m)$ but $D^\eta(p_m)(x) = \left(\frac{2^m}{A_1}\right)^d \left(\frac{2^m}{A_1}\right)^{|\eta|} D^\eta p\left(\frac{2^m x}{A_1}\right)$. The change of variables implies that $\|D^\eta(p_m)\|_1 = \left(\frac{2^m}{A_1}\right)^{|\eta|} \|D^\eta p\|_1 \leq C A_1^{-|\eta|} 2^{m|\eta|}$ when $|\eta| \leq 2$.

So we choose $A_1 \in \mathbb{N}$ such that $C A_1^{-|\eta|} \leq 1$ when $1 \leq |\eta| \leq 2$ and we have $\|D^\eta(\psi_m)\|_\infty \leq \|h\|_\infty \|D^\eta(p_m)\|_1 \leq 2^{m|\eta|}$ (when $\eta = 0$ the inequality is trivial). We take $A = A_1 + 1$.

We will define f by induction: $f_{-1} = 0$, $f_0 = \psi_0, \dots, f_{k+1} = f_k + \psi_{k+1}(1 - f_k)$. So

$$1 - f_{k+1} = 1 - f_k - \psi_{k+1}(1 - f_k) = (1 - \psi_{k+1})(1 - f_k) = (1 - \psi_{k+1})(1 - \psi_k) \dots (1 - \psi_0).$$

We let $1 - f = (1 - \psi_0) \dots (1 - \psi_{k'})$ where k' is the biggest natural number such that there exists a cube in Ω with side length $2^{-k'}$. We define $b_i = p_{t_i} * \mathbf{1}_{\tilde{Q}_i}(1 - f_{t_i-1})$, $i = 1, \dots, n$ where $2^{-t_i} = l(Q_i)$. It is easy to see that $f = \sum_{i=0}^n b_i$ and $\text{supp}(b_i) \subset AQ_i$, $i = 1, \dots, n$. Since $\psi_m(x) = 1$ when $x \in \cup_{Q \in F_m} Q$ it follows that $f|_{\cup_{i=1}^n Q_i} = 1$ and

$$|D^\eta(f_k)| = |D^\eta(1 - f_k)| = \left| \sum_{\eta_1 + \eta_2 + \dots + \eta_k = \eta} \prod_{j=1}^k D^{\eta_j}(1 - \psi_j) \right|$$

where $\eta_1, \eta_2, \dots, \eta_k$ are multi-indices. So $\eta_1 \in \mathbb{N}^d, \dots, \eta_k \in \mathbb{N}^d$. It follows from equation 3.3 and an elementary combinatorial argument that

$$|D^\eta(1 - f_k)| \leq \sum_{\eta_1 + \dots + \eta_k = \eta} \prod_{j=1}^k 2^{k|\eta_j|} = 2^{k|\eta|} \sum_{\eta_1 + \dots + \eta_k = \eta} 1 \leq C 2^{k|\eta|} (k+1)^{|\eta|}, \quad |\eta| \leq 2.$$

Hence

$$\begin{aligned} \|D^\eta(b_i)\|_\infty &= \left\| \sum_{Q \in \tilde{Q}_i} D^\eta((1 - f_{t_i-1})(p_{t_i} * \mathbf{1}_Q)) \right\|_\infty \\ &\leq C \sum_{\omega \leq \eta} \|D^\omega(1 - f_{t_i-1})\| \|D^{\eta-\omega}(p_{t_i} * \mathbf{1}_{Q_i})\|_\infty \\ &\leq C \sum_{\omega \leq \eta} 2^{(t_i-1)|\omega|} t_i^{|\eta|} 2^{t_i|\eta-\omega|} \\ &\leq C t_i^{|\eta|} 2^{t_i|\eta|}, \quad |\eta| \leq 2 \end{aligned}$$

□

Based on the previous proposition, we prove

Theorem 3.2.4. *Suppose K is a compact subset of \mathbb{R}^d , $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $0 < \alpha < 2$. If there exists $\gamma \in \mathbb{R}$ such that $\gamma < \beta$ where $\beta = (\alpha - 2)p' + d$, $(\frac{1}{p} + \frac{1}{p'} = 1)$ and $m_\gamma(K) = 0$ then K is $\Lambda_{\alpha,loc}^{p,q}$ -removable.*

Proof. We observe that A is a fixed number as in the previous proposition. Let us consider arbitrary $\epsilon > 0$. Since $m_\gamma(K) = 0$ there exists a covering of K by dyadic disjoint cubes Q_1, Q_2, \dots, Q_n such that $\sum_{i=1}^n 2^{-\gamma t_i} < \epsilon$ where $2^{-t_i} = l(Q_i)$, $t_i \in \mathbb{N}$, $i = 1, \dots, n$ (Recall that Q is a d -dimensional cube and $l(Q)$ is its side length). By the Proposition 3.2.2 there exists $f = \sum_{i=1}^n b_i$ such that:

$$(i) f|_K = 1;$$

$$(ii) \text{supp}(b_i) \subset AQ_i, i = 1, \dots, n,$$

$$(iii) \|D^\eta(b_i)\|_\infty \leq C 2^{|\eta|t_i} t_i^{|\eta|}, i = 1, \dots, n \text{ for } |\eta| \leq 2.$$

We define $s_i = \frac{t_i^2}{2^{t_i(d/p' - 2 + \alpha)}} (1 + \sqrt{d})^{M+2}$ (M is as in definition 1.3.10) and we define a_i by the formula

$$s_i a_i(x) = b_i(x), i = 1, \dots, n.$$

Then it is easy to see that:

$$(i) \text{supp}(a_i) \subset AQ;$$

$$(ii) |D^\eta a_i(x)|_\infty \leq c 2^{t_i(\frac{d}{p'} - (2 - \alpha) + |\eta|)} (1 + 2^{t_i}|x - x_{Q_i}|)^{-M-2} \text{ when } |\eta| \leq 2.$$

Since $\max\left(\left[d\left(\frac{1}{p} - 1\right)_+ - \alpha\right], -1\right) = -1$ when $0 < \alpha < 2$ it follows that $a_i, i = 1, \dots, n$ is a $(2 - \alpha, p')$ -molecule and we can use theorem 1.3.6 to estimate $\|f\|_{p', p', 2 - \alpha}^{p'} =$

$\|\sum_{i=1}^n s_i a_i\|_{p',p',2-\alpha}^{p'}$ as follows:

$$\begin{aligned}
\|f\|_{p',p',2-\alpha}^{p'} &\leq C \sum_{i=1}^n |s_i|^{p'} \\
&= C \sum_{i=1}^n \left(t_i^{2p'} 2^{-t_i(d+(\alpha-2)p')} \right) \\
&= C \sum_{i=1}^n \left(t_i^{2p'} 2^{-t_i(\beta-\gamma)} 2^{-\gamma t_i} \right) \\
&\leq C \sum_{i=1}^n 2^{-\gamma t_i} \\
&\leq C\epsilon.
\end{aligned}$$

In proving this inequality we are using the fact that without loss of generality we may assume $l(Q_i) = 2^{-t_i}, i = 1, \dots, n$ so small that $t_i^{2p'} 2^{-t_i(\beta-\gamma)} < 1$ since $\beta > \gamma$. By using theorem 3.1.3 we finish the proof. \square

3.3 Summary

The following questions need to be investigated: Suppose $K \subset \mathbb{R}^d$, is a compact, $0 < \alpha < 2$, $1 \leq p \leq \infty$, and $m_\beta(K) = 0$. In this case, is it true that K is $\Lambda_{\alpha,loc}^{p,q}$ -removable? Or if $m_\beta(K) > 0$ is it true that K is not $\Lambda_{\alpha,loc}^{p,q}$ -removable?

In case when $\alpha > 2 + \frac{p}{d}$ we know complete answer and when $2 \leq \alpha \leq 2 + \frac{p}{d}$ we only know a partial answer. The following questions still remain open: Suppose $m_d(K) > 0$ (d is the dimension of the Euclidean space where K is located). What can we say about $\Lambda_{\alpha,loc}^{p,q}$ -removability of K ? How does the exact answer to this question depend on the parameters α , p , and q ?

It seems that the Hausdorff measure is not giving exact characterization of K to answer those questions and we might need more subtle measure-geometric characterization of K .

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VITA

David Kighuradze

Candidate for the Degree of

Doctor of Philosophy

Dissertation: REMOVABLE SETS FOR HARMONIC FUNCTIONS IN BESOV SPACES

Major Field: Mathematics

Biographical:

Born in Tbilisi, Republic of Georgia, July 16, 1962.

Education:

Received the B.S. degree from Tbilisi State University, Tbilisi, Georgia (former soviet Union), 1984, in Mathematics

Received the M.S. degree from Tbilisi State University, Tbilisi, Georgia (former soviet Union), 1988, in Mathematics

Completed the requirements for the degree of Doctor of Philosophy with a major in Mathematics Oklahoma State University in July, 2009.

Experience:

Employed by Department of Mathematics, Tbilisi State University, as assistant Professor from July 1991 to December 1999; Employed by Department of Mathematics, Oklahoma State University, as a teaching assistant from January 2000 to July 2009.